Conservation laws on complex networks

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Abstract

This paper considers a system described by a conservation law on a general network and deals with solutions to Cauchy problems. The main application is to vehicular traffic, for which we refer to the Lighthill-Whitham-Richards (LWR) model. Assuming to have bounds on the conserved quantity, we are able to prove existence of solutions to Cauchy problems for every initial datum in L^1_{loc}. Moreover Lipschitz continuous dependence of the solution with respect to initial data is discussed.

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1 Introduction

Various fluido dynamic models were developed in the literature in order to describe the evolution of vehicular traffic in roads. They treat traffic from

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a macroscopic point of view: just the evolution of macroscopic variables, such as density and average velocity of cars, is considered. The Lighthill-Whitham-Richards (LWR) model (see [33, 36]), introduced in the 50s, is the prototype. It is based on the conservation of the number of cars and it consists of a single partial differential equation in conservation form.
From 1975 several second order models, i.e. models with two equations, were considered, see for example [1, 13, 24, 26, 35, 37, 40, 41], while a third order model was presented in [27]. An extension to multipopulation can be found; see [7]. We refer the reader to [6, 23, 28] for a general presentation of the various models.
More recently, a growing attention was devoted to extensions of the same models to networks; see for instance [4, 11, 12, 20, 21, 22, 29, 30, 31]. The interest was also motivated by other applications: data networks [18], supply chain [17, 25], air traffic management [5], gas pipelines [2, 14, 15]. Here we focus on the LWR model on a road network, but the results are of use to other research domains.

The main interest is in the Cauchy problem for a complex network. In some previous papers [12, 18, 20, 23], existence of weak entropic solutions was proved only for networks with junctions with at most two incoming and two outgoing roads and some specific dynamics at nodes.
Our construction is based on the wave-front tracking method; see [9, 16, 32]. More precisely, first we consider Riemann problems at nodes, which are Cauchy problems with constant initial data on each road. Notice that the only conservation of cars is not sufficient to determine a unique solution. Thus one has to prescribe solutions for every initial data and we call the relative map a *Riemann solver* at nodes. Then it is possible to construct approximate solutions using classical self-similar entropic solutions for Riemann problems inside roads and an assigned Riemann solver at junctions. As usual, the approach relies on three estimates: the number of waves, the number of wave interactions and total variation of the solution. While these estimates are straightforward on a real line (see [9]), they becomes difficult to be proved on complex networks (see [23]). In particular one has to rely on estimates on the total variation of the flux of the solution.

We provide a general strategy to overcome the technical problems: three key properties of Riemann solvers are defined (see Definitions 3.6-3.8), which guarantee the needed bounds and thus the existence of solutions to Cauchy problems. Our approach is valid for general networks, with no limitation on the type of junctions: in particular we extend all results of the literature. The main technical novelty is to get bounds on the total variation (in space) of solution flux via bounds on the positive variation (in time) of incoming fluxes at junctions.

To prove the validity of our approach, we show that the three key properties are shared by various Riemann solvers proposed in the literature. In particular, we consider three different kind of solutions at J, which we call Riemann solver $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$. The Riemann solver $\mathcal{RS}_1$ was proposed for vehicular traffic in [12]. It prescribes first a fixed distribution of traffic in outgoing roads, and then the maximization of the flux through the junction. The Riemann solver $\mathcal{RS}_2$ was introduced for data networks in [18]: first one maximizes the flux through the junction and then prescribes a distribution of traffic. The Riemann solver $\mathcal{RS}_3$ models car traffic at T-junctions; see [34]. Thanks to finite velocity of waves, one can reduce to treat the case of a single junction, with roads of infinite length.

The continuous dependence of solutions with respect to initial data is an open problem in the case of Riemann solver $\mathcal{RS}_1$. We remark that, in general, the Lipschitz continuous dependence with respect to initial data does not hold; see [12, 23]. As regards the Riemann solver $\mathcal{RS}_2$, we prove the Lipschitz continuous dependence with respect to initial conditions, by viewing L^1 as a Finsler manifold and considering "generalized tangent vectors". This method was proposed by Bressan [8] and improved in [10].

The paper is organized as follows. Section 2 contains the main definitions and notations. Section 3 deals with Riemann problems at the junction J, while the Riemann solvers $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$ are analyzed in Section 4. In Section 5 there are the statements of the main result about existence of solutions to Cauchy problems in the network, while in Section 5.1 the wave-front method is briefly described. Section 5.2 contains the technical proof about existence, while Section 6 deals with the Lipschitz continuous dependence of the solution with respect to initial conditions. Finally an appendix contains some technical results.

2 Basic Definitions and Notations

A complex networks is formed by a collection of roads and junctions. However, relying on finite velocity of waves, one can reduce to consider Cauchy problems for single junctions; see Theorem 4.3.9 of [23]. Thus, from now on, for sake of simplicity, we focus on a single junction with roads of infinite length.

Consider a junction J with n incoming roads $I_1, \ldots, I_n$ and m outgoing roads $I_{n+1}, \ldots, I_{n+m}$. We model each incoming road I_i ($i \in \{1, \ldots, n\}$) of the junction with the real interval $I_i =]-\infty, 0]$. Similarly we model each outgoing road I_j ($j \in \{n+1, \ldots, n+m\}$) of the junction with the real interval $I_j = [0, +\infty[$. On each road I_l ($l \in \{1, \ldots, n+m\}$) we consider the partial differential equation

$$(\rho_l)_t + f(\rho_l)_x = 0, \tag{1}$$

where $\rho_l = \rho_l(t,x) \in [0, \rho_{max}]$, is the *density* of cars, $v_l = v_l(\rho_l)$ is the *velocity* of cars and $f(\rho_l) = v_l(\rho_l)\,\rho_l$ is the *flux*. Hence the datum is given by a finite collection of functions ρ_l defined on $[0,+\infty[\times I_l$. For simplicity, we put $\rho_{max} = 1$.

On the flux f we make the following assumption

($\mathcal{F}$) $f : [0,1] \to \mathbb{R}$ is a Lipschitz continuous and concave function satisfying

1. $f(0) = f(1) = 0$;
2. there exists a unique $\sigma \in]0,1[$ such that f is strictly increasing in $[0,\sigma[$ and strictly decreasing in $]\sigma,1]$.

Remark 1 *Nonconcave flux functions were considered in the literature; see [28, 38]. The results of the paper can be generalized to those cases, provided 1. and 2. of ($\mathcal{F}$) are granted. This generalization is quite technical, since each Riemann problem inside roads may produce many waves (see e.g. [9, 23]). However, the main technical results remain valid. Finally, for sake of simplicity, we prefer to restrict ourselves to the concave case.*

The definitions of entropic solutions on roads and weak solutions at junctions are as follows.

Definition 2.1 *A function $\rho_l \in C([0,+\infty[; L^1_{loc}(I_l))$ is an entropy-admissible solution to (1) in the road I_l if the following holds.*

1. *For every function $\varphi : [0,+\infty[\times I_l \to \mathbb{R}$ smooth with compact support in $]0,+\infty[\times (I_l \setminus \{0\})$*

$$\int_0^{+\infty}\int_{I_l}\Big(\rho_l\frac{\partial\varphi}{\partial t} + f(\rho_l)\frac{\partial\varphi}{\partial x}\Big)dxdt = 0. \tag{2}$$

2. *For every $k \in \mathbb{R}$ and every $\tilde{\varphi} : [0,+\infty[\times I_l \to \mathbb{R}$ smooth, positive with compact support in $]0,+\infty[\times (I_l \setminus \{0\})$*

$$\int_0^{+\infty}\int_{I_l}\Big(|\rho_l - k|\frac{\partial\tilde{\varphi}}{\partial t} + \mathrm{sgn}(\rho_l - k)(f(\rho_l) - f(k))\frac{\partial\tilde{\varphi}}{\partial x}\Big)dxdt \geq 0. \tag{3}$$

Definition 2.2 *A collection of functions $\rho_l \in C([0,+\infty[; L^1_{loc}(I_l))$, $(l \in \{1,\ldots,n+m\})$ is a weak solution at J if*

1. *for every $l \in \{1, \ldots, n+m\}$, the function ρ_l is an entropy-admissible solution to (1) in the road I_l;*

2. *for every $l \in \{1, \ldots, n+m\}$ and for a.e. $t > 0$, the function $x \mapsto \rho_l(t, x)$ has a version with bounded total variation;*

3. *for a.e. $t > 0$, it holds*

$$\sum_{i=1}^{n} f(\rho_i(t, 0-)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, 0+)), \tag{4}$$

where ρ_l stands for the version with bounded total variation of 2.

We now define a set of matrices to describe solutions at junctions. First consider the set

$$\mathcal{A} := \left\{ A = \{a_{ji}\}_{\substack{i=1,\ldots,n \\ j=n+1,\ldots,n+m}} : \begin{array}{c} 0 < a_{ji} < 1 \ \forall i, j, \\ \sum\limits_{j=n+1}^{n+m} a_{ji} = 1 \ \forall i \end{array} \right\}. \tag{5}$$

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. For every $i = 1, \ldots, n$, we denote $H_i = \{e_i\}^\perp$. If $A \in \mathcal{A}$, then we write, for every $j = n+1, \ldots, n+m$, $a_j = (a_{j1}, \ldots, a_{jn}) \in \mathbb{R}^n$ and $H_j = \{a_j\}^\perp$. Let $\mathcal{K}$ be the set of indices $\mathbf{k} = (k_1, ..., k_\ell)$, $1 \leq \ell \leq n-1$, such that $0 \leq k_1 < k_2 < \cdots < k_\ell \leq n+m$ and for every $\mathbf{k} \in \mathcal{K}$ define

$$H_{\mathbf{k}} = \bigcap_{h=1}^{\ell} H_{k_h}.$$

Writing $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and following [12] we define the set

$$\mathfrak{N} := \left\{ A \in \mathcal{A} : \mathbf{1} \notin H_{\mathbf{k}}^\perp \text{ for every } \mathbf{k} \in \mathcal{K} \right\}. \tag{6}$$

Notice that, if $n > m$, then $\mathfrak{N} = \emptyset$. The matrices of $\mathfrak{N}$ will give rise to a unique solution to Riemann problems at J.

For later use, define also the set

$$\Theta = \left\{ \boldsymbol{\theta} = (\theta_1, \ldots, \theta_{n+m}) \in \mathbb{R}^{n+m} : \begin{array}{c} \theta_1 > 0, \cdots, \theta_{n+m} > 0, \\ \sum_{i=1}^{n} \theta_i = \sum_{j=n+1}^{n+m} \theta_j = 1 \end{array} \right\}. \tag{7}$$

3 The Riemann Problem

Fix a matrix $A \in \mathfrak{N}$, a vector $\boldsymbol{\theta} \in \Theta$ and $\rho_{1,0}, \ldots, \rho_{n+m,0} \in [0,1]$. Consider the Riemann problem at J

$$\begin{cases} \frac{\partial}{\partial t}\rho_l + \frac{\partial}{\partial x} f(\rho_l) = 0, \\ \rho_l(0,\cdot) = \rho_{0,l}, \end{cases} \quad l \in \{1, \ldots, n+m\}. \tag{8}$$

Remark 2 *The Riemann problem (8) can be interpreted as a collection of initial-boundary value problems, one for each road, with coupling conditions. Concerning this type of problems for conservation laws, we refer to [3] and to [19] for general theory.*
Conditions 2. and 3. of Definition 3.2 ensure that, on each road, an admissible solution to the corresponding initial-boundary value problem is achieved. See also Remark 3 below.

A solution to the Riemann problem at J is defined following Definition 2.2, i.e.

Definition 3.1 *A solution to the Riemann problem* (8) *is a weak solution at* J*, in the sense of Definition 2.2, such that* $\rho_l(0,x) = \rho_{l,0}$ *for every* $l \in \{1, \ldots, n+m\}$ *and for a.e.* $x \in I_l$*.*

We are now ready to introduce the key concept of Riemann solver at J.

Definition 3.2 *A Riemann solver* $\mathcal{RS}$ *is a function*

$$\begin{array}{rccc} \mathcal{RS}: & [0,1]^{n+m} & \longrightarrow & [0,1]^{n+m} \\ & (\rho_{1,0}, \ldots, \rho_{n+m,0}) & \longmapsto & (\bar\rho_1, \ldots, \bar\rho_{n+m}) \end{array}$$

satisfying the following

1. $\sum_{i=1}^{n} f(\bar\rho_i) = \sum_{j=n+1}^{n+m} f(\bar\rho_j)$;
2. *for every* $i \in \{1, \ldots, n\}$*, the classical Riemann problem*

$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = \begin{cases} \rho_{i,0}, & \text{if } x < 0, \\ \bar\rho_i, & \text{if } x > 0, \end{cases} \end{cases}$$

is solved with waves with negative speed;

3. for every $j \in \{n+1,\ldots,n+m\}$, the classical Riemann problem

$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \in \mathbb{R},\, t > 0, \\ \rho(0,x) = \begin{cases} \bar{\rho}_j, & \text{if } x < 0, \\ \rho_{j,0}, & \text{if } x > 0, \end{cases} \end{cases}$$

is solved with waves with positive speed.

Remark 3 *By Definition 3.2, a Riemann solver produces a solution to the Riemann problem (8), which conserves the number of cars at J and which generates waves with negative speed in incoming roads and waves with positive speed in outgoing roads.*

To effectively describe a solution to Riemann problems at J, a Riemann solver needs to satisfy the following consistency condition:

Definition 3.3 *We say that a Riemann solver $\mathcal{RS}$ satisfies the consistency condition if*

$$\mathcal{RS}(\mathcal{RS}(\rho_{1,0},\ldots,\rho_{n+m,0})) = \mathcal{RS}(\rho_{1,0},\ldots,\rho_{n+m,0})$$

for every $(\rho_{1,0},\ldots,\rho_{n+m,0}) \in [0,1]^{n+m}$.

Now we can state the three key properties of a Riemann solver, which will ensure the necessary bounds on approximate solutions (via wave-front tracking) and thus the existence of solutions to Cauchy problems. First we need some additional notation.

Definition 3.4 *We say that $(\rho_{1,0},\ldots,\rho_{n+m,0})$ is an equilibrium for the Riemann solver $\mathcal{RS}$ if*

$$\mathcal{RS}(\rho_{1,0},\ldots,\rho_{n+m,0}) = (\rho_{1,0},\ldots,\rho_{n+m,0}).$$

Definition 3.5 *We say that a datum $\rho_i \in [0,1]$ in an incoming road is a good datum if $\rho_i \in [\sigma,1]$ and a bad datum otherwise.*

We say that a datum $\rho_j \in [0,1]$ in an outgoing road is a good datum if $\rho_i \in [0,\sigma]$ and a bad datum otherwise.

The first property requires that equilibria are determined only by bad data values, more precisely:

Definition 3.6 *We say that a Riemann solver $\mathcal{RS}$ has the property (P1) if the following condition holds. Given $(\rho_{1,0},\ldots,\rho_{n+m,0})$ and $(\rho'_{1,0},\ldots,\rho'_{n+m,0})$ two initial data such that $\rho_{l,0} = \rho'_{l,0}$ whenever either $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum, then*

$$\mathcal{RS}(\rho_{1,0},\ldots,\rho_{n+m,0}) = \mathcal{RS}(\rho'_{1,0},\ldots,\rho'_{n+m,0}). \tag{9}$$

The second property asks for bounds in the increase of the flux variation for waves interacting with J. More precisely the latter should be bounded in terms of the strength of the interacting wave as well as the variation in the incoming fluxes.

Definition 3.7 *We say that a Riemann solver $\mathcal{RS}$ has the property (P2) if there exists a constant $C \geq 1$ such that, for every equilibrium $(\rho_{1,0}, \ldots, \rho_{n+m,0})$, the following two conditions are satisfied.*

1. *For every $i \in \{1, \ldots, n\}$ and for every $\rho_i \in [0,1]$ such that the wave $(\rho_i, \rho_{i,0})$ has positive speed, we have*

$$\sum_{\substack{l=1 \\ l \neq i}}^{n+m} |f(\hat{\rho}_l) - f(\rho_{l,0})| + |f(\hat{\rho}_i) - f(\rho_i)| - |f(\rho_{i,0}) - f(\rho_i)|$$
$$\leq C \min \left\{ |f(\rho_{i,0}) - f(\rho_i)|, \left| \sum_{l=1}^{n} [f(\hat{\rho}_l) - f(\rho_{l,0})] \right| \right\},$$

   where

$$(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) = \mathcal{RS}\left(\rho_{1,0}, \ldots, \rho_{i-1,0}, \rho_i, \rho_{i+1,0}, \ldots, \rho_{n+m,0}\right).$$

2. *For every $j \in \{n+1, \ldots, n+m\}$ and for every $\rho_j \in [0,1]$ such that the wave $(\rho_{j,0}, \rho_j)$ has negative speed, we have*

$$\sum_{\substack{l=1 \\ l \neq j}}^{n+m} |f(\hat{\rho}_l) - f(\rho_{l,0})| + |f(\hat{\rho}_j) - f(\rho_j)| - |f(\rho_{j,0}) - f(\rho_j)|$$
$$\leq C \min \left\{ |f(\rho_{j,0}) - f(\rho_j)|, \left| \sum_{l=1}^{n} [f(\hat{\rho}_l) - f(\rho_{l,0})] \right| \right\},$$

   where

$$(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) = \mathcal{RS}\left(\rho_{1,0}, \ldots, \rho_{j-1,0}, \rho_j, \rho_{j+1,0}, \ldots, \rho_{n+m,0}\right).$$

Finally, we state the third property: a wave interacting with J and provoking a flux decrease on a specific road should also gives rise to a decrease in the incoming fluxes.

Definition 3.8 *We say that a Riemann solver $\mathcal{RS}$ has the property (P3) if, for every equilibrium $(\rho_{1,0}, \ldots, \rho_{n+m,0})$, the following conditions are satisfied.*

1. *For every* $i \in \{1, \ldots, n\}$ *and for every* $\rho_i \in [0,1]$ *such that the wave* $(\rho_i, \rho_{i,0})$ *has positive speed and* $f(\rho_i) < f(\rho_{i,0})$*, we have*

$$\sum_{l=1}^{n} f(\rho_{l,0}) \geq \sum_{l=1}^{n} f(\hat{\rho}_l), \tag{10}$$

where

$$(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) = \mathcal{RS}\left(\rho_{1,0}, \ldots, \rho_{i-1,0}, \rho_i, \rho_{i+1,0}, \ldots, \rho_{n+m,0}\right).$$

2. *For every* $j \in \{n+1, \ldots, n+m\}$ *and for every* $\rho_j \in [0,1]$ *such that the wave* $(\rho_{j,0}, \rho_j)$ *has negative speed and* $f(\rho_j) < f(\rho_{j,0})$*, we have*

$$\sum_{l=1}^{n} f(\rho_{l,0}) \geq \sum_{l=1}^{n} f(\hat{\rho}_l), \tag{11}$$

where

$$(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) = \mathcal{RS}\left(\rho_{1,0}, \ldots, \rho_{j-1,0}, \rho_j, \rho_{j+1,0}, \ldots, \rho_{n+m,0}\right).$$

4 Riemann solvers

In this section we present some different Riemann solvers for the Riemann problem (8), proposed in recent literature. We verify for all of them the three key properties stated in the previous section.

Let us first illustrate some common facts to all Riemann solvers. Introduce the following sets and notations

1. for every $i \in \{1, \ldots, n\}$ define

$$\Omega_i = \begin{cases} [0, f(\rho_{i,0})], & \text{if } 0 \leq \rho_{i,0} \leq \sigma, \\ [0, f(\sigma)], & \text{if } \sigma \leq \rho_{i,0} \leq 1; \end{cases} \tag{12}$$

2. for every $j \in \{n+1, \ldots, n+m\}$ define

$$\Omega_j = \begin{cases} [0, f(\sigma)], & \text{if } 0 \leq \rho_{j,0} \leq \sigma, \\ [0, f(\rho_{j,0})], & \text{if } \sigma \leq \rho_{j,0} \leq 1; \end{cases} \tag{13}$$

3. for every $l \in \{1, \ldots, n+m\}$ denote

$$\gamma_l^{max} = \max \Omega_l. \tag{14}$$

For a flux satisfying $(\mathcal{F})$, we define:

Definition 4.1 *Let $\tau : [0,1] \to [0,1]$ be the map such that:*

1. *$f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0,1]$;*
2. *$\tau(\rho) \neq \rho$ for every $\rho \in [0,1] \setminus \{\sigma\}$.*

Clearly, the function τ is well defined and satisfies

$$0 \leq \rho \leq \sigma \Longleftrightarrow \sigma \leq \tau(\rho) \leq 1, \qquad \sigma \leq \rho \leq 1 \Longleftrightarrow 0 \leq \tau(\rho) \leq \sigma.$$

Then we can state the following:

Proposition 4.1 *The following statements hold.*

1. *For every $i \in \{1, \ldots, n\}$, an element $\bar{\gamma}$ belongs to Ω_i if and only if there exists $\bar{\rho}_i \in [0,1]$ such that $f(\bar{\rho}_i) = \bar{\gamma}$ and point 2 of Definition 3.2 is satisfied.*
2. *For every $j \in \{n+1, \ldots, n+m\}$, an element $\bar{\gamma}$ belongs to Ω_j if and only if there exists $\bar{\rho}_j \in [0,1]$ such that $f(\bar{\rho}_j) = \bar{\gamma}$ and point 3 of Definition 3.2 is satisfied.*

Proof. From 2. of Definition 3.2, $\bar{\rho}_i \in \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1]$ if $\rho_{i,0} < \sigma$, while $\bar{\rho}_i \in [\sigma, 1]$ otherwise. By definition of Ω_i, the first statement follows.
Similarly, by 3. of Definition 3.2, $\bar{\rho}_j \in \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[$ if $\rho_{j,0} > \sigma$, while $\bar{\rho}_j \in [0, \sigma]$ otherwise. By definition of Ω_j, the second statement follows. $\square$

4.1 Riemann Solver $\mathcal{RS}_1$

In this subsection, we consider the Riemann solver introduced for vehicular traffic in [12]. The construction can be summarized as follows.

1. Fix a matrix $A \in \mathfrak{N}$ and consider the closed, convex and not empty set
$$\Omega = \left\{ (\gamma_1, \cdots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i : A \cdot (\gamma_1, \cdots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} \Omega_j \right\}. \tag{15}$$
2. Find the point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \in \Omega$ which maximizes the function
$$E(\gamma_1, \ldots, \gamma_n) = \gamma_1 + \cdots + \gamma_n, \tag{16}$$
and define $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m})^T := A \cdot (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)^T$. Since $A \in \mathfrak{N}$, the point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ is uniquely defined.

3. For every $i \in \{1, \dots, n\}$, set $\bar{\rho}_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \sigma$. For every $j \in \{n+1, \dots, n+m\}$, set $\bar{\rho}_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \sigma$. Finally, define $\mathcal{RS}_1 : [0,1]^{n+m} \to [0,1]^{n+m}$ by

$$\mathcal{RS}_1(\rho_{1,0}, \dots, \rho_{n+m,0}) = (\bar{\rho}_1, \dots, \bar{\rho}_n, \bar{\rho}_{n+1}, \dots, \bar{\rho}_{n+m}). \tag{17}$$

We now verify the consistency condition as well as properties (P1)-(P3).

Lemma 4.1 *The function defined in (17) satisfies the consistency condition*

$$\mathcal{RS}_1(\mathcal{RS}_1(\rho_{1,0}, \dots, \rho_{n+m,0})) = \mathcal{RS}_1(\rho_{1,0}, \dots, \rho_{n+m,0}) \tag{18}$$

for every $(\rho_{1,0}, \dots, \rho_{n+m,0}) \in [0,1]^{n+m}$.

For a proof, see [12, 23].

Proposition 4.2 *The Riemann solver $\mathcal{RS}_1$ satisfies property (P1).*

Proof. Fix two initial data $(\rho_{1,0}, \dots, \rho_{n+m,0})$ and $(\rho'_{1,0}, \dots, \rho'_{n+m,0})$ with the property that $\rho_{l,0} = \rho'_{l,0}$ whenever either $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum. For every $l \in \{1, \dots, n+m\}$, consider Ω_l and Ω'_l the sets (12)-(13) respectively for the initial data $(\rho_{1,0}, \dots, \rho_{n+m,0})$ and $(\rho'_{1,0}, \dots, \rho'_{n+m,0})$. We easily deduce that $\Omega_l = \Omega'_l$ for every $l \in \{1, \dots, n+m\}$. Indeed if $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad data, then $\rho_{l,0} = \rho'_{l,0}$ and so $\Omega_l = \Omega'_l$. If $\rho_{l,0}$ is a good datum, then also $\rho'_{l,0}$ is a good datum (and viceversa) and so $\Omega_l = \Omega'_l = [0, f(\sigma)]$. Consequently we have the thesis, since the solution depends only on these sets and on the matrix A. □

Lemma 4.2 *Fix an equilibrium $(\rho_{1,0}, \dots, \rho_{n+m,0})$ for $\mathcal{RS}_1$ and consider, for some $l \in \{1, \dots, n+m\}$, $\rho_l \in [0,1]$ such that the wave $(\rho_l, \rho_{l,0})$ has positive speed if $l \leq n$, while the wave $(\rho_{l,0}, \rho_l)$ has negative speed if $l > n$. There exists a constant $\tilde{C} \geq 1$ such that*

$$\sum_{\substack{h=1 \\ h \neq l}}^{n+m} |f(\hat{\rho}_h) - f(\rho_{h,0})| + |f(\hat{\rho}_l) - f(\rho_l)| \leq \tilde{C} \, |f(\rho_l) - f(\rho_{l,0})| , \tag{19}$$

where

$$(\hat{\rho}_1, \dots, \hat{\rho}_{n+m}) = \mathcal{RS}_1(\rho_{1,0}, \dots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \dots, \rho_{n+m,0}).$$

PROOF. Denote with Ω^- and with Ω the sets, defined in (15), respectively for the initial data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ and $(\rho_{1,0}, \ldots, \rho_l, \ldots, \rho_{n+m,0})$. It is easy to see that, by construction, $\Omega^- \subseteq \Omega$ or $\Omega \subseteq \Omega^-$. We have two different possibilities:

1. $\max_{(\gamma_1,\ldots,\gamma_n)\in\Omega^-} E(\gamma_1, \ldots, \gamma_n) = \max_{(\gamma_1,\ldots,\gamma_n)\in\Omega} E(\gamma_1, \ldots, \gamma_n)$,
2. $\max_{(\gamma_1,\ldots,\gamma_n)\in\Omega^-} E(\gamma_1, \ldots, \gamma_n) \neq \max_{(\gamma_1,\ldots,\gamma_n)\in\Omega} E(\gamma_1, \ldots, \gamma_n)$,

where E is the linear function defined in (16). If

$$\max_{(\gamma_1,\ldots,\gamma_n)\in\Omega^-} E(\gamma_1, \ldots, \gamma_n) = \max_{(\gamma_1,\ldots,\gamma_n)\in\Omega} E(\gamma_1, \ldots, \gamma_n),$$

then, since $A \in \mathfrak{N}$, there exists a unique

$$(\bar\gamma_1, \ldots, \bar\gamma_n) = (f(\rho_{1,0}), \ldots, f(\rho_{n,0})) \in \Omega \cap \Omega^-$$

such that

$$E(\bar\gamma_1, \ldots, \bar\gamma_n) = \max_{(\gamma_1,\ldots,\gamma_n)\in\Omega^-} E(\gamma_1, \ldots, \gamma_n) = \max_{(\gamma_1,\ldots,\gamma_n)\in\Omega} E(\gamma_1, \ldots, \gamma_n).$$

Therefore there is only one wave, produced by $\mathcal{RS}_1$ at J, in the road I_l. Hence

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} |f(\hat\rho_h) - f(\rho_{h,0})| + |f(\hat\rho_l) - f(\rho_l)| = |f(\hat\rho_l) - f(\rho_l)| = |f(\rho_{l,0}) - f(\rho_l)|$$

and the conclusion follows.

Consider the other case, i.e.

$$\max_{(\gamma_1,\ldots,\gamma_n)\in\Omega^-} E(\gamma_1, \ldots, \gamma_n) \neq \max_{(\gamma_1,\ldots,\gamma_n)\in\Omega} E(\gamma_1, \ldots, \gamma_n).$$

Denote with $(\gamma_1^-, \ldots, \gamma_n^-) \in \Omega^-$ and with $(\bar\gamma_1, \ldots, \bar\gamma_n) \in \Omega$ the points of maximum of E respectively on Ω^- and on Ω. We have that $(\gamma_1^-, \ldots, \gamma_n^-) = (f(\rho_{1,0}), \ldots, f(\rho_{n,0}))$, $(\gamma_1^-, \ldots, \gamma_n^-) \in \partial\Omega^-$ and $(\bar\gamma_1, \ldots, \bar\gamma_n) \in \partial\Omega$. Since the directions of the faces of Ω^- and Ω depend only on the coefficients of A and the difference between the two sets depends only by the variation of a single constraint, then there exists a constant $\bar C$ such that

$$\left|(\gamma_1^-, \ldots, \gamma_n^-) - (\bar\gamma_1, \ldots, \bar\gamma_n)\right| \leq \bar C \left|f(\rho_{l,0}) - f(\rho_l)\right|.$$

Hence

$$\begin{aligned}\sum_{\substack{h=1\\h\neq l}}^{n+m} &\left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| \\ &\leq \sum_{h=1}^{n+m} \left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| \\ \leq 2\bar{C}\sum_{i=1}^{n} &\left|\bar{\gamma}_i - \gamma_{i,0}\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| \leq (2\bar{C}+1)\left|f(\rho_{l,0}) - f(\rho_l)\right|\end{aligned}$$

and the conclusion follows. □

Proposition 4.3 *The Riemann solver $\mathcal{RS}_1$ satisfies property (P2).*

PROOF. Fix an equilibrium $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ for $\mathcal{RS}_1$ and $l \in \{1, \ldots, n+m\}$. Assume $l \leq n$, $\rho_l \in [0, 1]$ is such that the wave $(\rho_l, \rho_{l,0})$ has positive speed, the other case being similar. Define

$$(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) = \mathcal{RS}(\rho_{1,0}, \ldots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \ldots, \rho_{n+m,0}).$$

Lemma 4.2 implies that

$$\begin{aligned}\sum_{\substack{h=1\\h\neq l}}^{n+m} \left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| - \left|f(\rho_{l,0}) - f(\rho_l)\right| \\ \leq (\tilde{C}-1)\left|f(\rho_{l,0}) - f(\rho_l)\right|.\end{aligned}$$

Call $\Gamma^- = \sum_{i=1}^{n} f(\rho_{i,0})$ and $\Gamma^+ = \sum_{i=1}^{n} f(\hat{\rho}_i)$. Since the direction of the faces of the set Ω, defined in (15), depend only on the matrix $A \in \mathfrak{N}$ and the solution for the flux lies on the boundary of Ω, we have that $|\Gamma^- - \Gamma^+|$ is proportional to

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} \left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| - \left|f(\rho_{l,0}) - f(\rho_l)\right|$$

and so the conclusion follows. □

Proposition 4.4 *The Riemann solver $\mathcal{RS}_1$ satisfies property (P3).*

PROOF. Fix an equilibrium $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ for $\mathcal{RS}_1$ and $l \in \{1, \ldots, n+m\}$. Consider just the case $l \leq n$, the other case being similar. Assume that $\rho_l \in [0,1]$ is such that the wave $(\rho_l, \rho_{l,0})$ has positive speed and $f(\rho_l) < f(\rho_{l,0})$. Define

$$(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) = \mathcal{RS}(\rho_{1,0}, \ldots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \ldots, \rho_{n+m,0}).$$

The Rankine-Hugoniot condition implies that $\rho_l < \rho_{l,0}$ and so ρ_l is a bad datum. Call Ω^- and Ω^+ respectively the sets (15) for the initial data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ and $(\rho_{1,0}, \ldots, \rho_l, \ldots, \rho_{n+m,0})$. Since ρ_l is a bad datum and $f(\rho_l) < f(\rho_{l,0})$, then $\Omega^+ \subseteq \Omega^-$ and so

$$\sum_{i=1}^{n} f(\rho_{i,0}) \geq \sum_{i=1}^{n} f(\hat{\rho}_i).$$

The proof is finished. □

4.2 Riemann Solver $\mathcal{RS}_2$

In this subsection, we consider the Riemann solver, introduced in [18] for data networks; see also [23]. The construction consists of the following steps.

1. Fix $\boldsymbol{\theta} \in \Theta$ and define

   $$\Gamma_{inc} = \sum_{i=1}^{n} \sup \Omega_i, \quad \Gamma_{out} = \sum_{j=n+1}^{n+m} \sup \Omega_j,$$

   then the maximal possible through-flow at the crossing is

   $$\Gamma = \min \{\Gamma_{inc}, \Gamma_{out}\} \ .$$

2. Introduce the closed, convex and not empty sets

   $$\begin{aligned} I &= \left\{ (\gamma_1, \ldots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i \colon \sum_{i=1}^{n} \gamma_i = \Gamma \right\} \\ J &= \left\{ (\gamma_{n+1}, \ldots, \gamma_{n+m}) \in \prod_{j=n+1}^{n+m} \Omega_j \colon \sum_{j=n+1}^{n+m} \gamma_j = \Gamma \right\} . \end{aligned}$$

3. Denote with $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ the orthogonal projection on the convex set I of the point $(\Gamma\theta_1, \ldots, \Gamma\theta_n)$ and with $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m})$ the orthogonal projection on the convex set J of the point $(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m})$.

4. For every $i \in \{1, \ldots, n\}$, define $\bar{\rho}_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \sigma$. For every $j \in \{n+1, \ldots, n+m\}$, define $\bar{\rho}_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \sigma$. Finally, define $\mathcal{RS}_2 : [0,1]^{n+m} \to [0,1]^{n+m}$ by

$$\mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0}) = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}) . \tag{20}$$

The following result holds.

Lemma 4.3 *The function defined in (20) satisfies the consistency condition*

$$\mathcal{RS}_2(\mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0})) = \mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0}) \tag{21}$$

for every $(\rho_{1,0}, \ldots, \rho_{n+m,0}) \in [0,1]^{n+m}$.

PROOF. Consider $(\rho_{1,0}, \ldots, \rho_{n+m,0}) \in [0,1]^{n+m}$, call $\Gamma_{0,inc}$, $\Gamma_{0,out}$, Γ_0 the numbers defined in 1. of $\mathcal{RS}_2$ and call I_0 and J_0 the sets defined in 2. of $\mathcal{RS}_2$. Let

$$(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}) = \mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0})$$

and

$$(\bar{\gamma}_1, \ldots, \bar{\gamma}_{n+m}) = (f(\bar{\rho}_1), \ldots, f(\bar{\rho}_{n+m})).$$

Similarly to above, call Γ_{inc}, Γ_{out}, Γ, the numbers defined in 1. and I and J the sets defined in 2. with respect to the initial condition $(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m})$. In order to prove (21), we need to consider the following possibilities.

1. $\Gamma_0 = \Gamma_{0,inc} \leq \Gamma_{0,out}$.
2. $\Gamma_0 = \Gamma_{0,out} \leq \Gamma_{0,inc}$.

We restrict to the first case, since the second one is completely symmetric. For every $i \in \{1, \ldots, n\}$, $\bar{\rho}_i \in \{\rho_{i,0}, \sigma\}$. More precisely, if $\rho_{i,0} < \sigma$, then $\bar{\rho}_i = \rho_{i,0}$, while if $\rho_{i,0} \geq \sigma$, then $\bar{\rho}_i = \sigma$.

Applying $\mathcal{RS}_2$ to the point $(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m})$, we deduce $\Gamma_{inc} = \Gamma_{0,inc}$, $\Gamma_{out} \geq \Gamma_{0,out}$, $\Gamma = \Gamma_0$, $I = I_0$, and $J_0 \subseteq J$. More precisely, $\Gamma_{out} > \Gamma_{0,out}$ if and only if there exists $j \in \{n+1, \ldots, n+m\}$ such that $\rho_{j,0} > \sigma$ and $\bar{\rho}_j \leq \sigma$. Define

$$\tilde{A} = \{j \in \{n+1, \ldots, n+m\} : \rho_{j,0} > \sigma, \, \bar{\rho}_j \leq \sigma\}$$

and $\tilde{B} = \{n+1, \ldots, n+m\} \setminus \tilde{A}$. We easily deduce that the projection of $(\Gamma_0\theta_1, \ldots, \Gamma_0\theta_n)$ on I_0 is the same as the projection of $(\Gamma\theta_1, \ldots, \Gamma\theta_n)$ on I. We also claim that the projection of $(\Gamma_0\theta_{n+1}, \ldots, \Gamma_0\theta_{n+m})$ on J_0 is the same as the projection of $(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m})$ on J. In fact, if $J = J_0$, then the claim is obvious. Assume therefore that $J_0 \subsetneqq J$. If we denote with P_C the orthogonal projection on a closed and convex subset C of $\mathbb{R}^m$, then

$$(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m}) = P_{J_0}(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m}).$$

Therefore, if we choose a point $(x_{n+1}, \ldots, x_{n+m}) \in J_0$, then the scalar product

$$(\Gamma\theta_{n+1} - \bar{\gamma}_{n+1}, \ldots, \Gamma\theta_{n+m} - \bar{\gamma}_{n+m}) \cdot (x_{n+1} - \bar{\gamma}_{n+1}, \ldots, x_{n+m} - \bar{\gamma}_{n+m}) \leq 0.$$

Notice that $J \setminus J_0$ is given by points $(\gamma_{n+1}, \ldots, \gamma_{n+m})$ satisfying

$$f(\rho_{j,0}) < \gamma_j \leq f(\sigma)$$

for some $j \in \tilde{A}$. Since $\bar{\gamma}_j < f(\rho_{j,0})$ for every $j \in \tilde{A}$, then for every point $(\tilde{x}_{n+1}, \ldots, \tilde{x}_{n+m})$ of J such that $\tilde{x}_j > \bar{\gamma}_j$ for some $j \in \tilde{A}$, there exist $\zeta > 0$ and a point $(x_{n+1}, \ldots, x_{n+m}) \in J_0$ such that $x_j > \bar{\gamma}_j$ for some $j \in \tilde{A}$ and

$$(\tilde{x}_{n+1} - \bar{\gamma}_{n+1}, \ldots, \tilde{x}_{n+m} - \bar{\gamma}_{n+m}) = \zeta(x_{n+1} - \bar{\gamma}_{n+1}, \ldots, x_{n+m} - \bar{\gamma}_{n+m}).$$

This fact permits to conclude that

$$(\Gamma\theta_{n+1} - \bar{\gamma}_{n+1}, \ldots, \Gamma\theta_{n+m} - \bar{\gamma}_{n+m}) \cdot (\tilde{x}_{n+1} - \bar{\gamma}_{n+1}, \ldots, \tilde{x}_{n+m} - \bar{\gamma}_{n+m}) \leq 0$$

for every $(\tilde{x}_{n+1}, \ldots, \tilde{x}_{n+m}) \in J$ and so

$$(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m}) = P_J(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m}).$$

This concludes the proof. □

Before proving (P1)-(P3), we need to prove some technical lemmas about projections.

Lemma 4.4 *Fix $N \in \mathbb{N} \setminus \{0\}$, a set $\mathcal{P} = \prod_{l=1}^{N}[0, a_l]$, where $a_l > 0$ for every $l \in \{1, \ldots, N\}$, and an N-dimensional vector $(\vartheta_1, \ldots, \vartheta_N)$ such that $\vartheta_l > 0$ for every $l \in \{1, \ldots, N\}$ and $\sum_{l=1}^{N} \vartheta_l = 1$. For $0 \leq \Lambda \leq \sum_{l=1}^{N} a_l$, denote with $(\zeta_1, \ldots, \zeta_N) = P_{\mathcal{I}}(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$ the orthogonal projection of $(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_1)$ on the set*

$$\mathcal{I} = \left\{ (\gamma_1, \ldots, \gamma_N) \in \mathcal{P} : \sum_{l=1}^{N} \gamma_l = \Lambda \right\}.$$

Then the value ζ_l ($l \in \{1, \ldots, N\}$) depends on Λ in a continuous way. Moreover, for all but a finite number of $0 < \Lambda < \sum_{l=1}^{N} a_l$, the derivative of ζ_l with respect to Λ exists and satisfies $\frac{\partial}{\partial\Lambda}\zeta_l \geq 0$.

Proof. The continuity of ζ_l w.r.t. Λ is trivial. The differentiability of ζ_l w.r.t. Λ is instead granted for all values of Λ such that locally the projection $P_{\mathcal{I}}(\Lambda\vartheta_1, \dots, \Lambda\vartheta_N)$ either is $(\Lambda\vartheta_1, \dots, \Lambda\vartheta_N)$ or lyes in the same face of $\mathcal{I}$. By linearity, this happens for all but a finite number of values of Λ. Thus we are left with last statement.
The conclusion is evident if $P_{\mathcal{I}}(\Lambda\vartheta_1, \dots, \Lambda\vartheta_N)$ is equal to $(\Lambda\vartheta_1, \dots, \Lambda\vartheta_N)$. So assume that

$$P_{\mathcal{I}}(\Lambda\vartheta_1, \dots, \Lambda\vartheta_N) \neq (\Lambda\vartheta_1, \dots, \Lambda\vartheta_N),$$

i.e. $(\zeta_1, \dots, \zeta_N)$ belongs to the boundary of $\mathcal{I}$. Moreover the case $N = 1$ is trivial, so we consider $N \geq 2$.

Since $\vartheta_k > 0$ for every $k \in \{1, \dots, N\}$, then $\zeta_k > 0$ for every $k \in \{1, \dots, N\}$. Assume, for simplicity, that there exists $\bar{k} \in \{1, \dots, N-1\}$, such that

$$\zeta_k = a_k,$$

for every $k = \bar{k}+1, \dots, N$, and $\zeta_k < a_k$ otherwise. The vector $(\zeta_1, \dots, \zeta_N)$ can be written in the form

$$\Lambda(\vartheta_1, \dots, \vartheta_N) + t(v_1, \dots, v_N),$$

where $t > 0$, $(v_1, \dots, v_N)$ depends on Λ and on a_k and it satisfies $\sum_{l=1}^N v_l = 0$. Hence, for every $k = \bar{k}+1, \dots, N$, we deduce that

$$t = \frac{a_k - \vartheta_k\Lambda}{v_k}$$

and

$$v_k = \frac{a_k - \vartheta_k\Lambda}{a_N - \vartheta_N\Lambda} v_N.$$

Since the projection minimizes the distance, in order to find $(\zeta_1, \dots, \zeta_n)$, it is sufficient to minimize t (or equivalently to maximize v_N^2) under the constraints

$$v_k = \frac{a_k - \vartheta_k\Lambda}{a_N - \vartheta_N\Lambda} v_N, \qquad k = \bar{k}+1, \dots, N, \tag{22}$$

$$\|(v_1, \dots, v_N)\|^2 = 1, \tag{23}$$

$$\sum_{l=1}^N v_l = 0. \tag{24}$$

We apply the Lagrangian multiplier method to maximize v_N^2 under the constraints (22)-(24). For simplicity, define $f = v_N^2$, $g_k = v_k - \frac{a_k - \vartheta_k\Lambda}{a_N - \vartheta_N\Lambda} v_N$ for

$k = \bar{k}+1, \ldots, N$, $g_{N+1} = \sum_{l=1}^{N} v_l$ and finally $g_{N+2} = \|(v_1, \ldots, v_N)\|^2 - 1$. So we deal with the critical points of the function

$$f + \sum_{k=\bar{k}+1}^{N+2} \lambda_k g_k,$$

depending on the variables $(v_1, \ldots, v_N)$, where the coefficients λ_k belong to $\mathbb{R}$. Differentiating the previous function with respect to v_i $(i = 1, \ldots, \bar{k})$, we find that

$$\lambda_{N+1} + 2\lambda_{N+2} v_i = 0,$$

which implies that $v_1 = \cdots = v_{\bar{k}} = \bar{v}$ for some $\bar{v} \neq 0$, since λ_{N+1} and λ_{N+2} are non trivial. Thus equations (22) and (24) imply that

$$\bar{v} = -\frac{A v_N}{\bar{k}},$$

where

$$A = 1 + \sum_{k=\bar{k}+1}^{N-1} \frac{a_k - \vartheta_k \Lambda}{a_N - \vartheta_N \Lambda}.$$

Hence, for every $i = 1, \ldots, \bar{k}$,

$$\begin{aligned}
\frac{\partial}{\partial \Lambda}(\zeta_i) &= \frac{\partial}{\partial \Lambda}(\Lambda \vartheta_i + t v_i) = \frac{\partial}{\partial \Lambda}\left(\Lambda \vartheta_i - \frac{a_N - \vartheta_N \Lambda}{v_N} \cdot \frac{A v_N}{\bar{k}}\right) \\
&= \frac{\partial}{\partial \Lambda}\left(\Lambda \vartheta_i - \frac{a_N - \vartheta_N \Lambda}{\bar{k}} \cdot A\right) \\
&= \frac{\partial}{\partial \Lambda}\left(\Lambda \vartheta_i - \frac{1}{\bar{k}} \sum_{k=\bar{k}+1}^{N} (a_k - \vartheta_k \Lambda)\right) \\
&= \vartheta_i + \frac{1}{\bar{k}} \sum_{k=\bar{k}+1}^{N} \vartheta_k > 0,
\end{aligned}$$

while, for every $i = \bar{k}+1, \ldots, N$, we have $\frac{\partial}{\partial \Lambda}(\zeta_i) = 0$. □

Lemma 4.5 *Fix $N \in \mathbb{N} \setminus \{0\}$, a set $\mathcal{P} = \prod_{l=1}^{N} [0, a_l]$, where $a_l > 0$ for every $l \in \{1, \ldots, N\}$, and an N-dimensional vector $(\vartheta_1, \ldots, \vartheta_N)$ such that $\vartheta_l > 0$ for every $l \in \{1, \ldots, N\}$ and $\sum_{l=1}^{N} \vartheta_l = 1$. Fix $0 \leq \Lambda < \sum_{l=1}^{N} a_l$*

and denote with $(\zeta_1,\ldots,\zeta_N) = P_{\mathcal{I}}(\Lambda\vartheta_1,\ldots,\Lambda\vartheta_N)$ *the orthogonal projection of* $(\Lambda\vartheta_1,\ldots,\Lambda\vartheta_1)$ *on the set*

$$\mathcal{I} = \left\{ (\gamma_1,\ldots,\gamma_N) \in \mathcal{P} : \sum_{l=1}^{N} \gamma_l = \Lambda \right\}.$$

Then the value ζ_l *(*$l \in \{1,\ldots,N\}$*) depends in a continuous way on* a_h *for* $h \in \{1,\ldots,N\}$*. Moreover, if* $l \neq h$*, then for all but a finite number of* a_h *it is differentiable and it holds* $\frac{\partial \zeta_l}{\partial a_h} \leq 0$*.*

Proof. The continuity of ζ_l w.r.t. a_h is obvious. The differentiability of ζ_l w.r.t. a_h is instead granted for all values of a_h such that $\Lambda < \sum_{l=1}^{N} a_l$ and the projection $P_{\mathcal{I}}(\Lambda\vartheta_1,\ldots,\Lambda\vartheta_N)$ locally either is $(\Lambda\vartheta_1,\ldots,\Lambda\vartheta_N)$ or lyes in the same face of $\mathcal{I}$. The latter happens for all but a finite number of values of a_h by linearity, thus we are left with last statement.
The case $N = 1$ is trivial, hence we assume that $N \geq 2$ and $l, h \in \{1,\ldots,N\}$ with $l \neq h$. If $(\zeta_1,\ldots,\zeta_N)$ is equal to $\Lambda(\vartheta_1,\ldots,\vartheta_N)$, then the claim is obvious. Assume therefore that

$$(\zeta_1,\ldots,\zeta_N) \neq \Lambda(\vartheta_1,\ldots,\vartheta_N).$$

In this case $(\zeta_1,\ldots,\zeta_N)$ belongs to the topological boundary of $\mathcal{P}$ contained in the space $\sum_{i=1}^{n} \gamma_i = \Lambda$.
As in the proof of Lemma 4.4, we deduce $\zeta_k > 0$ for every $k \in \{1,\ldots,N\}$ and assume there exists $\bar{k} \in \{1,\ldots,N-1\}$, such that $\zeta_k = a_k$, for every $k = \bar{k}+1,\ldots,N$, and $\zeta_k < a_k$ otherwise. Again (see the proof of Lemma 4.4) we write $(\zeta_1,\ldots,\zeta_N) = \Lambda(\vartheta_1,\ldots,\vartheta_N) + t(v_1,\ldots,v_N)$, and deduce $v_1 = \cdots = v_{\bar{k}} = \bar{v}$ for some $\bar{v} \neq 0$.
Now, notice that $\frac{\partial}{\partial a_h}(\zeta_i) = 0$ if $i \geq \bar{k}+1$ and $i \neq h$. While, if $i \leq \bar{k}$, then

$$\frac{\partial}{\partial a_h}(\zeta_i) = \frac{\partial}{\partial a_h}(\Lambda\vartheta_i + t v_i) = \frac{\partial}{\partial a_h}(t\bar{v}),$$

since Λ is fixed. Thus $\frac{\partial}{\partial a_h}(\zeta_i)$ is independent from i and, finally, the equation $\sum_{i=1}^{n} \zeta_i = \Lambda$ implies that

$$\frac{\partial}{\partial a_h}(\zeta_i) \leq 0;$$

so the proof is finished. □

Remark 4 *Note that, in Lemma 4.4, we assume that every* a_l *(*$l \in \{1,\ldots,N\}$*) is fixed and that the coefficient* Λ *varies.*
On the contrary, in Lemma 4.5, we assume that Λ *is fixed and that the coefficients* a_l *vary.*

Proposition 4.5 *The Riemann solver $\mathcal{RS}_2$ satisfies property (P1).*

PROOF. Fix two initial data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ and $(\rho'_{1,0}, \ldots, \rho'_{n+m,0})$ with the property that $\rho_{l,0} = \rho'_{l,0}$ whenever either $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum. For every $l \in \{1, \ldots, n+m\}$, consider Ω_l and Ω'_l the sets (12)-(13) respectively for the initial data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ and $(\rho'_{1,0}, \ldots, \rho'_{n+m,0})$. With the same considerations of the proof of Proposition 4.2, we deduce that $\Omega_l = \Omega'_l$ for every $l \in \{1, \ldots, n+m\}$. Hence we have the thesis. □

Lemma 4.6 *Fix an equilibrium $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ for $\mathcal{RS}_2$ and consider, for some $l \in \{1, \ldots, n+m\}$, $\rho_l \in [0,1]$ such that the wave $(\rho_l, \rho_{l,0})$ has positive speed if $l \le n$, while the wave $(\rho_{l,0}, \rho_l)$ has negative speed if $l > n$. Then*

$$\sum_{\substack{h=1 \\ h \neq l}}^{n+m} |f(\hat\rho_h) - f(\rho_{h,0})| + |f(\hat\rho_l) - f(\rho_l)| = |f(\rho_l) - f(\rho_{l,0})|, \qquad (25)$$

where

$$(\hat\rho_1, \ldots, \hat\rho_{n+m}) = \mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \ldots, \rho_{n+m,0}).$$

PROOF. In this proof we use the following notation.

- Γ^-_{inc}, Γ^-_{out}, Γ^-, I^- and J^- denote the numbers and the sets defined in points 1 and 2 of Subsection 4.2 for the initial condition $(\rho_{1,0}, \ldots, \rho_{n+m,0})$.
- Γ^+_{inc}, Γ^+_{out}, Γ^+, I^+ and J^+ denote the numbers and the sets defined in points 1 and 2 of Subsection 4.2 for the initial condition $(\hat\rho_1, \ldots, \hat\rho_{n+m})$.
- $\bar\Gamma_{inc}$, $\bar\Gamma_{out}$, $\bar\Gamma$, $\bar I$ and $\bar J$ denote the numbers and the sets defined in points 1 and 2 of Subsection 4.2 for the initial condition $(\rho_{1,0}, \ldots, \rho_l, \ldots, \rho_{n+m,0})$.
- $\Omega_{h,0}$ denotes the set defined in (12) or in (13) with respect to $\rho_{h,0}$ for $h \in \{1, \ldots, n+m\}$.
- $\overline{\Omega}_l$ denotes the set defined in (12) or in (13) with respect to ρ_l.

Notice that $\bar\Gamma = \Gamma^+$. We have the following two possibilities.

1. $\Gamma^-_{inc} \le \Gamma^-_{out}$.
2. $\Gamma^-_{inc} > \Gamma^-_{out}$.

We deal only with the proof of the first case, since the second one can be treated in the same way. Assume, therefore, $\Gamma^-_{inc} \le \Gamma^-_{out}$. In this case $\Gamma^- = \Gamma^-_{inc}$ and $\rho_{i,0} \le \sigma$ for every $i \in \{1, \ldots, n\}$. There are two different situations: $l \le n$ and $l > n$.

Assume first $l \le n$. We noticed that $\rho_{l,0} \le \sigma$ and so $\rho_l < \sigma$, since the speed of the wave is positive. We have

$$\bar{\Gamma}_{inc} = \Gamma^-_{inc} - f(\rho_{l,0}) + f(\rho_l) \tag{26}$$

and

$$\bar{\Gamma}_{out} = \Gamma^-_{out}.$$

If $\bar{\Gamma}_{inc} \le \bar{\Gamma}_{out}$, then no wave is produced in incoming roads and at most m waves are produced in outgoing roads. The total variation of the flux due to these waves is

$$\sum_{j=n+1}^{n+m} \left| f(\hat{\rho}_j) - f(\rho_{j,0}) \right|.$$

Therefore

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} \left| f(\hat{\rho}_h) - f(\rho_{h,0}) \right| + \left| f(\hat{\rho}_l) - f(\rho_l) \right| - \left| f(\rho_l) - f(\rho_{l,0}) \right|$$

$$= -\left| f(\rho_l) - f(\rho_{l,0}) \right| + \sum_{j=n+1}^{n+m} \left| f(\rho_{j,0}) - f(\hat{\rho}_j) \right|.$$

If $f(\rho_l) < f(\rho_{l,0})$, then $\bar{\Gamma} < \Gamma^-$ and the sets $\bar{J} \subseteq J^-$ differ only for the values of $\bar{\Gamma}$, Γ^-, since the wave $(\rho_l, \rho_{l,0})$ does not affect $\Omega_{j,0}$ for every $j \in \{n+1, , \ldots, n+m\}$; hence we apply Lemma 4.4 and deduce that $f(\hat{\rho}_j) \le f(\rho_{j,0})$ for every $j \in \{n+1, \ldots, n+m\}$.
If instead $f(\rho_l) > f(\rho_{l,0})$, then $\bar{\Gamma} > \Gamma^-$ and, with similar considerations as the previous ones, we have that $f(\hat{\rho}_j) \ge f(\rho_{j,0})$ for every $j \in \{n+1, \ldots, n+m\}$. Therefore, we have

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} \left| f(\hat{\rho}_h) - f(\rho_{h,0}) \right| + \left| f(\hat{\rho}_l) - f(\rho_l) \right| - \left| f(\rho_l) - f(\rho_{l,0}) \right|$$

$$= \operatorname{sgn}(f(\rho_l) - f(\rho_{l,0})) \cdot \left(-f(\rho_l) + f(\rho_{l,0}) + \sum_{j=n+1}^{n+m} \left(f(\hat{\rho}_j) - f(\rho_{j,0}) \right) \right)$$
$$= \operatorname{sgn}(f(\rho_l) - f(\rho_{l,0})) \left(-f(\rho_l) + f(\rho_{l,0}) + \Gamma^+ - \Gamma^- \right)$$
$$= \operatorname{sgn}(f(\rho_l) - f(\rho_{l,0})) \left(-f(\rho_l) + f(\rho_{l,0}) + \bar{\Gamma} - \Gamma^-_{inc} \right)$$
$$= \operatorname{sgn}(f(\rho_l) - f(\rho_{l,0})) \left(-f(\rho_l) + f(\rho_{l,0}) + \bar{\Gamma}_{inc} - \Gamma^-_{inc} \right) = 0\,,$$

where we used equation (26) and the equality $\bar{\Gamma} = \Gamma^+$. Thus the conclusion follows in the case $\bar{\Gamma}_{inc} \leq \bar{\Gamma}_{out}$.

If $\bar{\Gamma}_{inc} > \bar{\Gamma}_{out}$, then $\bar{\Gamma} = \Gamma^+ = \bar{\Gamma}_{out} = \Gamma^+_{out}$ and $\bar{\Gamma}_{inc} > \bar{\Gamma} \geq \Gamma^-_{inc}$. Moreover we deduce that $f(\rho_l) > f(\rho_{l,0})$ and so the total variation of the flux due to the interacting wave is, in this case, equal to

$$|f(\rho_l) - f(\rho_{l,0})| = f(\rho_l) - f(\rho_{l,0}).$$

Since $\bar{\Gamma} = \bar{\Gamma}_{out}$, then in the outgoing roads there is the formation of at most m waves and the trace of the flux of the solution at the junction is the maximum possible. This implies that $f(\hat{\rho}_j) \geq f(\rho_{j,0})$ for every $j \in \{n+1, \dots, n+m\}$. Therefore the total variation of the flux in outgoing roads after the interaction produced at J is given by $\bar{\Gamma} - \Gamma^-_{inc}$.
By $\bar{\Gamma}_{inc} > \bar{\Gamma}$, in the incoming roads there is the production of at most n waves. In this case, the trace of the solution in an incoming road is a good datum (see Definition 3.5), since $\rho_{i,0} \leq \sigma$ for every $i \in \{1, \dots, n\}$, $\rho_l < \sigma$ and the speed of the produced waves is negative. Then $f(\rho_{h,0}) \geq f(\hat{\rho}_h)$ for every $h \in \{1, \dots, n\}$, $h \neq l$ and $f(\hat{\rho}_l) \leq f(\rho_l)$.
Without loss of generality, we may assume that the interacting wave is in the road I_1, i.e. $l = 1$; hence the total variation of the flux due to the waves is

$$\begin{aligned}
&\sum_{\substack{h=1 \\ h\neq l}}^{n+m} |f(\hat{\rho}_h) - f(\rho_{h,0})| + |f(\hat{\rho}_l) - f(\rho_l)| - |f(\rho_l) - f(\rho_{l,0})| \\
&= \sum_{h=2}^{n+m} |f(\hat{\rho}_h) - f(\rho_{h,0})| + |f(\hat{\rho}_1) - f(\rho_1)| - |f(\rho_1) - f(\rho_{1,0})| \\
&= \sum_{i=2}^{n} [f(\rho_{i,0}) - f(\hat{\rho}_i)] + \bar{\Gamma} - \Gamma^-_{inc} + f(\rho_1) - f(\hat{\rho}_1) - f(\rho_1) + f(\rho_{1,0}) \\
&= \sum_{i=1}^{n} [f(\rho_{i,0})] - \sum_{i=1}^{n} [f(\hat{\rho}_i)] + \bar{\Gamma} - \Gamma^-_{inc} \\
&= \sum_{i=1}^{n} [f(\rho_{i,0})] - \Gamma^+ + \bar{\Gamma} - \Gamma^-_{inc} \\
&= \Gamma^-_{inc} - \Gamma^+ + \bar{\Gamma} - \Gamma^-_{inc} = 0.
\end{aligned}$$

Thus the conclusion follows provided $\bar{\Gamma}_{inc} > \bar{\Gamma}_{out}$. Therefore the case $l \leq n$ is completed.

Assume now $l > n$ and, without loss of generality, $l = n+1$. We consider three different situations.

If $\Gamma_{out}^- \leq \bar{\Gamma}_{out}$, then $\bar{\Gamma} = \bar{\Gamma}_{inc} = \Gamma_{inc}^-$, and so nothing happens in incoming roads. If $\Gamma_{out}^- = \bar{\Gamma}_{out}$, then both $\rho_{n+1,0}$ and ρ_{n+1} are good data and this is not possible by the velocity of the wave. Since the wave $(\rho_{n+1,0}, \rho_{n+1})$ has negative speed, then $\Gamma_{out}^- < \bar{\Gamma}_{out}$. The only possibility is that $\rho_{n+1,0}$ is a bad datum, $\rho_{n+1} \in [\sigma, \rho_{n+1,0}[$ and so $f(\rho_{n+1}) > f(\rho_{n+1,0})$. Moreover, since the wave $(\hat{\rho}_{n+1}, \rho_{n+1})$ has positive speed, then $f(\rho_{n+1}) \geq f(\hat{\rho}_{n+1})$. Therefore

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} |f(\hat{\rho}_h) - f(\rho_{h,0})| + |f(\hat{\rho}_l) - f(\rho_l)| - |f(\rho_l) - f(\rho_{l,0})|$$

$$= \sum_{j=n+2}^{n+m} |f(\rho_{j,0}) - f(\hat{\rho}_j)| + |f(\hat{\rho}_{n+1}) - f(\rho_{n+1})| - |f(\rho_{n+1}) - f(\rho_{n+1,0})|$$

$$= \sum_{j=n+2}^{n+m} |f(\rho_{j,0}) - f(\hat{\rho}_j)| + f(\rho_{n+1}) - f(\hat{\rho}_{n+1}) - f(\rho_{n+1}) + f(\rho_{n+1,0})$$

$$= \sum_{j=n+2}^{n+m} |f(\rho_{j,0}) - f(\hat{\rho}_j)| - f(\hat{\rho}_{n+1}) + f(\rho_{n+1,0}).$$

Since $\bar{\Gamma} = \Gamma^-$ and $\Omega_{l,0} \subseteq \overline{\Omega}_l$, we may apply Lemma 4.5 and deduce that $f(\rho_{j,0}) \geq f(\hat{\rho}_j)$ for every $j \in \{n+2, \ldots, n+m\}$ and so

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} |f(\hat{\rho}_h) - f(\rho_{h,0})| + |f(\hat{\rho}_l) - f(\rho_l)| - |f(\rho_l) - f(\rho_{l,0})|$$

$$= \sum_{j=n+1}^{n+m} [f(\rho_{j,0})] - \sum_{j=n+1}^{n+m} [f(\hat{\rho}_j)] = \Gamma^- - \Gamma^+ = 0$$

and so we have the thesis in the case $\Gamma_{out}^- \leq \bar{\Gamma}_{out}$.

If $\Gamma_{out}^- > \bar{\Gamma}_{out} \geq \Gamma_{inc}^-$, then $\rho_{n+1,0} < \rho_{n+1}$ and $f(\rho_{n+1}) < f(\rho_{n+1,0})$, since the wave $(\rho_{n+1,0}, \rho_{n+1})$ has negative speed. Thus $\bar{\Gamma} = \Gamma_{inc}^-$ and no wave is produced in incoming roads and also no wave is produced in the road I_{n+1}; i.e. $\hat{\rho}_{n+1} = \rho_{n+1}$. Moreover $\overline{\Omega}_l \subseteq \Omega_{l,0}$ and so, by Lemma 4.5, we have

$f(\rho_{j,0}) \leq f(\hat{\rho}_j)$ for every $j \in \{n+2, \ldots, n+m\}$. Therefore

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} |f(\hat{\rho}_h) - f(\rho_{h,0})| + |f(\hat{\rho}_l) - f(\rho_l)| - |f(\rho_l) - f(\rho_{l,0})|$$

$$= \sum_{j=n+2}^{n+m} |f(\hat{\rho}_j) - f(\rho_{j,0})| - |f(\rho_{n+1}) - f(\rho_{n+1,0})|$$

$$= \sum_{j=n+2}^{n+m} [f(\hat{\rho}_j) - f(\rho_{j,0})] + f(\rho_{n+1}) - f(\rho_{n+1,0})$$

$$\sum_{j=n+1}^{n+m} [f(\hat{\rho}_j) - f(\rho_{j,0})] = \bar{\Gamma} - \Gamma^- = 0$$

and so we have the thesis in the case $\Gamma^-_{out} > \bar{\Gamma}_{out} \geq \Gamma^-_{inc}$.

If $\Gamma^-_{out} > \bar{\Gamma}_{out}$ and $\bar{\Gamma}_{out} < \Gamma^-_{inc}$, then $\rho_{n+1,0} < \rho_{n+1}$ and $f(\rho_{n+1}) < f(\rho_{n+1,0})$, since the wave $(\rho_{n+1,0}, \rho_{n+1})$ has negative speed. Moreover $\bar{\Gamma} = \bar{\Gamma}_{out}$ and so no wave is produced in I_{n+1}, waves with decreasing flux are produced in incoming roads, and waves with increasing flux are produced in outgoing roads, i.e. $\hat{\rho}_{n+1} = \rho_{n+1}$, $f(\hat{\rho}_i) \leq f(\rho_{i,0})$ for every $i \in \{1, \ldots, n\}$ and $f(\hat{\rho}_j) \geq f(\rho_{j,0})$ for every $j \in \{n+2, \ldots, n+m\}$. Hence

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} |f(\hat{\rho}_h) - f(\rho_{h,0})| + |f(\hat{\rho}_l) - f(\rho_l)| - |f(\rho_l) - f(\rho_{l,0})|$$

$$= \sum_{i=1}^{n} |f(\hat{\rho}_i) - f(\rho_{i,0})| + \sum_{j=n+2}^{n+m} |f(\hat{\rho}_j) - f(\rho_{j,0})| - |f(\rho_{n+1,0}) - f(\rho_{n+1})|$$

$$= \sum_{i=1}^{n} [f(\rho_{i,0}) - f(\hat{\rho}_i)] + \sum_{j=n+2}^{n+m} [f(\hat{\rho}_j) - f(\rho_{j,0})] + f(\rho_{n+1}) - f(\rho_{n+1,0})$$

$$= \Gamma^- - \Gamma^+ + \sum_{j=n+1}^{n+m} [f(\hat{\rho}_j) - f(\rho_{j,0})] = \Gamma^- - \Gamma^+ + \Gamma^+ - \Gamma^- = 0,$$

and we have the thesis in the case $\Gamma^-_{out} > \bar{\Gamma}_{out}$ and $\bar{\Gamma}_{out} < \Gamma^-_{inc}$.

The proof is thus finished. □

Proposition 4.6 *The Riemann solver $\mathcal{RS}_2$ satisfies property (P2).*

PROOF. Fix an equilibrium $(\rho_{1,0}, \dots, \rho_{n+m,0})$ for $\mathcal{RS}_2$ and consider, for some $l \in \{1, \dots, n+m\}$, $\rho_l \in [0,1]$ such that the wave $(\rho_l, \rho_{l,0})$ has positive speed if $l \leq n$, while the wave $(\rho_{l,0}, \rho_l)$ has negative speed if $l > n$. Define

$$(\hat{\rho}_1, \dots, \hat{\rho}_{n+m}) = \mathcal{RS}_2(\rho_{1,0}, \dots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \dots, \rho_{n+m,0}).$$

By Lemma 4.6 we have

$$\sum_{\substack{h=1 \\ h \neq l}}^{n+m} |f(\hat{\rho}_h) - f(\rho_{h,0})| + |f(\hat{\rho}_l) - f(\rho_l)| - |f(\rho_{l,0}) - f(\rho_l)|$$
$$= |f(\rho_{l,0}) - f(\rho_l)| - |f(\rho_{l,0}) - f(\rho_l)| = 0$$

and so (P2) holds. □

Proposition 4.7 *The Riemann solver $\mathcal{RS}_2$ satisfies property (P3).*

PROOF. Fix an equilibrium $(\rho_{1,0}, \dots, \rho_{n+m,0})$ for $\mathcal{RS}_2$ and $l \in \{1, \dots, n+m\}$. Consider just the case $l \leq n$, the other case being similar. Assume that $\rho_l \in [0,1]$ is such that the wave $(\rho_l, \rho_{l,0})$ has positive speed and $f(\rho_l) < f(\rho_{l,0})$. Define

$$(\hat{\rho}_1, \dots, \hat{\rho}_{n+m}) = \mathcal{RS}(\rho_{1,0}, \dots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \dots, \rho_{n+m,0}).$$

The Rankine-Hugoniot condition implies that $\rho_l < \rho_{l,0}$ and so ρ_l is a bad datum. Call Γ^- and Γ^+ respectively the values, defined in point 1 of the procedure for $\mathcal{RS}_2$, for initial data $(\rho_{1,0}, \dots, \rho_{n+m,0})$ and $(\rho_{1,0}, \dots, \rho_l, \dots, \rho_{n+m,0})$. Since ρ_l is a bad datum, then $\Gamma^- \geq \Gamma^+$ and so

$$\sum_{i=1}^{n} f(\rho_{i,0}) \geq \sum_{i=1}^{n} f(\hat{\rho}_i).$$

The proof is finished. □

4.3 Riemann Solver $\mathcal{RS}_3$

In this subsection, we consider the Riemann solver, introduced in [34] to model T-junctions. Consider a junction J with n incoming and $m = n$ outgoing roads and fix a positive coefficient Γ_J, which is the maximum capacity of the junction. The construction can be done in the following way.

1. Fix $\boldsymbol{\theta} \in \Theta$. For every $i \in \{1, \ldots, n\}$, define

$$\Gamma_i = \min \left\{ \gamma_i^{max}, \gamma_{i+n}^{max} \right\}$$

where the numbers γ_l^{max} are defined in (14). Then the maximal possible through-flow at J is

$$\Gamma = \sum_{i=1}^{n} \Gamma_i.$$

2. Introduce the closed, convex and not empty set

$$I = \left\{ (\gamma_1, \ldots, \gamma_n) \in \prod_{i=1}^{n} [0, \Gamma_i] \colon \sum_{i=1}^{n} \gamma_i = \min \left\{ \Gamma, \Gamma_J \right\} \right\}.$$

3. Denote with $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ the orthogonal projection on the convex set I of the point $(\min\{\Gamma, \Gamma_J\}\theta_1, \ldots, \min\{\Gamma, \Gamma_J\}\theta_n)$ and set $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{2n}) = (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$.

4. For every $i \in \{1, \ldots, n\}$, define $\bar{\rho}_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \sigma$. For every $j \in \{n+1, \ldots, n+m\}$, define $\bar{\rho}_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \sigma$. Finally, define $\mathcal{RS}_3 : [0,1]^{n+m} \to [0,1]^{n+m}$ by

$$\mathcal{RS}_3(\rho_{1,0}, \ldots, \rho_{n+m,0}) = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}). \qquad (27)$$

The following result holds.

Lemma 4.7 *The function defined in (27) satisfies the consistency condition*

$$\mathcal{RS}_3(\mathcal{RS}_3(\rho_{1,0}, \ldots, \rho_{n+m,0})) = \mathcal{RS}_3(\rho_{1,0}, \ldots, \rho_{n+m,0}) \qquad (28)$$

for every $(\rho_{1,0}, \ldots, \rho_{n+m,0}) \in [0,1]^{n+m}$.

For a proof, see Proposition 2.4 of [34].

Proposition 4.8 *The Riemann solver $\mathcal{RS}_3$ satisfies property (P1).*

Proof. Fix two initial data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ and $(\rho'_{1,0}, \ldots, \rho'_{n+m,0})$ with the property that $\rho_{l,0} = \rho'_{l,0}$ whenever either $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum. For every $l \in \{1, \ldots, n+m\}$, consider Ω_l and Ω'_l the sets (12)-(13) respectively for the initial data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ and $(\rho'_{1,0}, \ldots, \rho'_{n+m,0})$. With the same considerations of the proof of Proposition 4.2, we deduce that $\Omega_l = \Omega'_l$ for every $l \in \{1, \ldots, n+m\}$. Hence we have the thesis. □

Proposition 4.9 *The Riemann solver $\mathcal{RS}_3$ satisfies properties (P2) and (P3).*

The proof is completely similar to the proofs of properties (P2) and (P3) for the Riemann solver $\mathcal{RS}_2$ and so omitted.

5 The Cauchy Problem

In this section, we deal with the Cauchy problem at the junction J. Fix n initial data for incoming roads $\rho_{1,0}, \ldots, \rho_{n,0} \in BV(]-\infty, 0]; [0,1])$ and m initial data for outgoing roads $\rho_{n+1,0}, \ldots, \rho_{n+m,0} \in BV([0, +\infty[; [0,1])$. Consider the Cauchy problem at J:

$$\begin{cases} \frac{\partial}{\partial t}\rho_l(t,x) + \frac{\partial}{\partial x} f(\rho_l(t,x)) = 0, & x \in I_l \setminus \{0\},\, t > 0, \\ \rho_l(0,x) = \rho_{0,l}(x), & x \in I_l, \end{cases} \qquad l \in \{1, \ldots, n+m\}. \tag{29}$$

The main result is the following theorem.

Theorem 5.1 *Consider the Cauchy problem (29) and a Riemann solver $\mathcal{RS}$ satisfying the consistency condition and the properties (P1), (P2) and (P3). Then there exists a weak solution at J $(\rho_1(t,x), \ldots, \rho_{n+m}(t,x))$ such that*

1. *for every $l \in \{1, \ldots, n+m\}$, $\rho_l(0,x) = \rho_{0,l}(x)$ for a.e. $x \in I_l$;*

2. *for a.e. $t > 0$,*

$$\mathcal{RS}(\rho_1(t,0-), \ldots, \rho_{n+m}(t,0+)) = (\rho_1(t,0-), \ldots, \rho_{n+m}(t,0+)). \tag{30}$$

The proof of the theorem is given in next sections. In [12, 18, 23, 34], existence of solutions was proved for Riemann solvers $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$ for a junction J with at most two incoming and two outgoing roads.

Remark 5 *Notice that, under the hypotheses of the paper, every weak solution at J $(\rho_1(t,x), \ldots, \rho_{n+m}(t,x))$ admits strong traces for a.e. $t > 0$ $(\rho_1(t,0-), \ldots, \rho_{n+m}(t,0+))$; see [39].*

5.1 Wave-Front Tracking

Since solutions to Riemann problems are given, we are able to construct piecewise constant approximations via wave-front tracking algorithm; see [9] for the general theory and [23] in the case of networks.

Definition 5.1 *Given $\varepsilon > 0$ and a Riemann solver $\mathcal{RS}$, we say that the map $\bar{\rho}_\varepsilon = (\bar{\rho}_{1,\varepsilon}, \ldots, \bar{\rho}_{n+m,\varepsilon})$ is an ε-approximate wave-front tracking solution to (29) with respect to $\mathcal{RS}$ if the following conditions hold.*

1. *For every $l \in \{1, \ldots, n+m\}$, $\bar{\rho}_{l,\varepsilon} \in C([0, +\infty[; L^1_{loc}(I_l; [0,1]))$.*

2. *For every $l \in \{1, \ldots, n+m\}$, $\bar{\rho}_{l,\varepsilon}(t,x)$ is piecewise constant, with discontinuities occurring along finitely many straight lines in the (t,x)-plane. Moreover jumps of $\bar{\rho}_{l,\varepsilon}(t,x)$ can be shocks or rarefactions and are indexed by $\mathcal{J}_l(t) = \mathcal{S}_l(t) \cup \mathcal{R}_l(t)$.*

3. *For every $l \in \{1, \ldots, n+m\}$, along each shock $x(t) = x_{l,\alpha}(t)$, $\alpha \in \mathcal{S}_l(t)$, we have*
$$\bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)-) < \bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)+).$$
Moreover
$$\left| \dot{x}_{l,\alpha}(t) - \frac{f(\bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)-)) - f(\bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)+))}{\bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)-) - \bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)+)} \right| \leq \varepsilon.$$

4. *For every $l \in \{1, \ldots, n+m\}$, along each rarefaction front $x(t) = x_{l,\alpha}(t)$, $\alpha \in \mathcal{R}_l(t)$, we have*
$$\bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)+) < \bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)-) < \bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)+) + \varepsilon.$$
Moreover
$$\dot{x}_{l,\alpha}(t) \in \left[f'(\bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)-)), f'(\bar{\rho}_{l,\varepsilon}(t, x_{l,\alpha}(t)+))\right].$$

5. *For every $l \in \{1, \ldots, n+m\}$,*
$$\|\bar{\rho}_{l,\varepsilon}(0, \cdot) - \rho_{0,l}(\cdot)\|_{L^1(I_l)} < \varepsilon.$$

6. *For a.e. $t > 0$*
$$\mathcal{RS}(\bar{\rho}_{1,\varepsilon}(t, 0-), \ldots, \bar{\rho}_{n+m,\varepsilon}(t, 0+)) = (\bar{\rho}_{1,\varepsilon}(t, 0-), \ldots, \bar{\rho}_{n+m,\varepsilon}(t, 0+)).$$

Fix a Riemann solver $\mathcal{RS}$ satisfying the consistency condition and the properties (P1), (P2) and (P3). For every $l \in \{1, \ldots, n+m\}$, consider a sequence $\rho_{0,l,\nu}$ of piecewise constant functions defined on I_l such that $\rho_{0,l,\nu}$ has a finite number of discontinuities and $\lim_{\nu \to +\infty} \rho_{0,l,\nu} = \rho_{0,l}$ in $L^1_{loc}(I_l; [0,1])$. For every $\nu \in \mathbb{N}$, we apply the following procedure. At time $t = 0$, we solve the Riemann problem at J (according to $\mathcal{RS}$) and all Riemann problems in each road. We approximate every rarefaction wave with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$ travelling with the Rankine-Hugoniot speed. Moreover, if σ is in the range of a rarefaction shock, then its speed is zero. We repeat the previous construction at every time at which interactions between waves or of waves with J happen.

Remark 6 *By slightly modifying the speed of waves, we may assume that, at every positive time t, at most one interaction happens. Moreover, at every interaction time, either two waves interact in a road or a wave reaches the junction J.*

Remark 7 *For interactions in roads, we split rarefaction waves into rarefaction fans just at time $t = 0$. At the junction J, instead, we allow the formation of rarefaction fans at every positive time.*

Let us introduce the concepts of generation order for waves and of big shocks. We need these definitions in the proof of existence of a wave-front tracking approximate solution.

Definition 5.2 *A wave of $\bar{\rho}_\varepsilon$, generated at time $t = 0$, is said an original wave or a wave with generation order 1.*

If a wave with generation order $k \geq 1$ interacts with J, then the produced waves are said of generation $k + 1$.

If a wave with generation order $k \geq 1$ interacts in a road with a wave with generation order $k' \geq 1$, then the produced wave is said of generation $\min\{k, k'\}$.

Definition 5.3 *We say that a wave (ρ_l, ρ_r) in a road is a big shock if $\rho_l < \sigma < \rho_r$.*

We have the following proposition.

Proposition 5.1 *For every $\nu \in \mathbb{N}$ the previous construction can be done for every positive time, producing an $\frac{1}{\nu}$-approximate wave-front tracking solution to (29) with respect to $\mathcal{RS}$.*

Proof. For every $l \in \{1, \ldots, n+m\}$ and every $\nu \in \mathbb{N}$, call $\rho_{l,\nu}$ the function built by the previous procedure. Moreover, for every $l \in \{1, \ldots, n+m\}$, $\nu \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{0\}$ and for every time $t \geq 0$, define the functions $N_{l,\nu}(t)$ and $M_{l,k,\nu}(t)$, which count respectively the number of discontinuities of $\rho_{l,\nu}(t, \cdot)$ and the number of waves with generation order k of $\rho_{l,\nu}(t, \cdot)$.

Assume, by contradiction, that, there exist $\bar{\nu} \in \mathbb{N}$ and $T > 0$ such that

$$\sum_{l=1}^{n+m} N_{l,\bar{\nu}}(t) < +\infty$$

for every $t \in [0, T[$, and

$$\limsup_{t \to T^-} \sum_{l=1}^{n+m} N_{l,\bar{\nu}}(t) = +\infty. \tag{31}$$

Note that, for every time t,

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(t) \leq \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(0+) < +\infty.$$

Indeed, $\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(t)$ is locally constant and can vary only at interaction times in the following way:

1. if at $\bar{t} > 0$ a wave with generation order 1 reaches the junction J, then

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-) - 1;$$

2. if at $\bar{t} > 0$ two waves with generation order 1 interact in a road, then

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-) - 1;$$

3. if at $\bar{t} > 0$ a wave with generation order k_1 interacts with a wave of order k_2 in a road with $k_1 + k_2 \geq 3$, then

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-).$$

Moreover, for every $l \in \{1, \ldots, n+m\}$ and for every $k \geq 0$, the function $M_{l,k,\bar{\nu}}(\cdot)$ is decreasing inside the roads. For every $k \in \mathbb{N} \setminus \{0\}$ and for every time $t > 0$, we have

$$\sum_{l=1}^{n+m} M_{l,k,\bar{\nu}}(t) \leq (K_{\bar{\nu}})^{k-1} \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(0+) = (K_{\bar{\nu}})^{k-1} \sum_{l=1}^{n+m} N_{l,\bar{\nu}}(0+) < +\infty,$$

where $K_{\bar{\nu}} = (n+m)\bar{\nu}$. This bound is due to the fact that each wave with generation order k can interact with J and produce at most $\bar{\nu}$ waves with generation order $k+1$ in each road (in the case of rarefactions).

Now, there exists $0 < \eta < T$ such that no wave with generation order 1 interacts with J in the time interval $(T-\eta, T)$. Equation (31) implies also that in $(T-\eta, T)$ there are an infinite number of interactions of waves with J. Since waves of generation order 1 do not interact in $(T-\eta, T)$, the only possibility is that a wave with generation order $k \geq 2$ comes back to J producing waves of order $k+1$, some of which come back to J producing waves of order $k+2$ and so on. Moreover by Lemma 4.3.7 of [23] (see the Appendix), if a wave of generation order $k \geq 2$, interacts with J from a road in $(T-\eta, T)$, then, after the interaction, the datum in that road is bad, since the wave can not interact with waves of generation order 1 and come back to J. In a road a bad datum at J can change only in the following cases:

1. an original wave interacts with J from the road;
2. a wave, which is a big shock, is originated at J on a road and the new datum at J is good.

Obviously in the time interval $(T-\eta, T)$ the first possibility can not happen; so only the second possibilities may happen. Assume that there exist $t_1, t_2 \in (T-\eta, T)$ with $t_1 < t_2$ such that a big shock is originated at J at time t_1 in a road and come back to J at time t_2. Then, in that road, the datum before t_1 and after t_2 is necessarily the same and it is bad. Thus every road I_l may take only a precise bad value $\bar{\rho}_l$, otherwise good values. The key point is that, at every time $t \in (T-\eta, T)$, there are finitely many possible combinations of bad data at the junction J (obtained choosing the roads which present a bad datum at J, the precise value being fixed). By property (P1) (i.e. the image of $\mathcal{RS}$ depends only on the values of bad data) we deduce that, for $t \in (T-\eta, T)$, $\rho_{\bar{\nu}}(t)$ at J may take only a finite number of values, thus waves produced by J have a finite set of possible velocities. Finally, interactions of waves with J can not accumulate at time T. This concludes the proof by contradiction. □

5.2 Existence of Solutions

This subsection is devoted to the estimate of the total variation of the flux along an approximate wave-front tracking solution and to the construction of solutions to the Cauchy problem. Fix an ε-approximate wave-front tracking solution to (29) $\bar{\rho}_\varepsilon$, in the sense of Definition 5.1, with respect to a Riemann solver $\mathcal{RS}$ satisfying the consistency condition and the properties (P1), (P2) and (P3).

Define the functionals

$$\Gamma(t) := \sum_{i=1}^{n} f(\bar{\rho}_{i,\varepsilon}(t, 0-)) \tag{32}$$

and

$$\text{Tot.Var.}_f(t) := \sum_{l=1}^{n+m} \text{Tot.Var.}\left(f\left(\bar{\rho}_{l,\varepsilon}(t,\cdot)\right)\right). \tag{33}$$

It is clear that these functionals are well defined for every positive time and can vary only when a wave reaches J or when two waves interact in a road. By definition we easily derive the bound

$$0 \leq \Gamma(t) \leq n f(\sigma) \tag{34}$$

for every $t \geq 0$.

Let us give a definition.

Definition 5.4 *We say that a wave* (ρ_l, ρ_r) *interacting with* J *from an incoming road has decreasing flux (resp. increasing flux) if* $f(\rho_l) < f(\rho_r)$ *(resp.* $f(\rho_l) > f(\rho_r)$*).*

We say that a wave (ρ_l, ρ_r) *interacting with* J *from an outgoing road has decreasing flux (resp. increasing flux) if* $f(\rho_l) > f(\rho_r)$ *(resp.* $f(\rho_l) < f(\rho_r)$*).*

Lemma 5.1 *The following statements hold.*

1. *Assume that a wave with decreasing flux, connecting* ρ_l *with* ρ_r*, reaches* J *from an incoming road. Then* (ρ_l, ρ_r) *is a shock wave and* ρ_l *is a bad datum.*

2. *Assume that a wave with decreasing flux, connecting* ρ_l *and* ρ_r*, reaches* J *from an outgoing road. Then* (ρ_l, ρ_r) *is a shock wave and* ρ_r *is a bad datum.*

PROOF. Let us consider an incoming road and a wave (ρ_l, ρ_r), which reaches the junction J with decreasing flux. The wave has positive speed and so $\rho_l < \rho_r$. Since f is decreasing in $[\sigma, 1]$ then $\rho_l < \sigma$. It means that the wave is a shock wave and ρ_l is a bad datum.

The situation for an outgoing road is completely symmetric. $\square$

Lemma 5.2 *Assume that a wave connecting ρ_l and ρ_r reaches the junction J at time $\bar{t} > 0$ with decreasing flux. Then*

$$\Gamma(\bar{t}+) \leq \Gamma(\bar{t}-). \tag{35}$$

PROOF. It is a simple consequence of property (P3) of the Riemann solver $\mathcal{RS}$. □

Corollary 5.1 *If a wave generated at J returns to J without interacting with waves with generation order* 1*, then it has decreasing flux and produces a decrease of* Γ.

PROOF. Consider a wave generated at J, which does not interact with waves with generation order 1. Since the network is composed by a single junction, then the speed of the wave can change only if the wave interacts with waves with generation order $k \geq 2$, i.e. with waves produced by J. Under these assumptions, the speed of the wave can change sign, only in the case the wave is a big shock or it interacts with a big shock; see Lemma 4.3.7 of [23] (see the Appendix). In any case, the wave is a big shock when it returns to J. Moreover it must have positive velocity if it is in an incoming road, while it must have negative velocity in the other case. Therefore it is a wave with decreasing flux and the conclusion easily follows by Lemma 5.2. □

Lemma 5.3 *Assume that a wave (ρ_l, ρ_r) interacts with J at a time $\bar{t} > 0$, then*

$$Tot.Var._f(\bar{t}+) \leq (C+1)\, Tot.Var._f(\bar{t}-). \tag{36}$$

where C is given by property (P2) of the Riemann solver $\mathcal{RS}$.

PROOF. By property (P2), we get

$$\text{Tot.Var.}_f(\bar{t}+) - \text{Tot.Var.}_f(\bar{t}-) \leq C\,|f(\rho_l) - f(\rho_r)|\,.$$

Therefore we have

$$\begin{aligned}
\text{Tot.Var.}_f(\bar{t}+) &\leq \text{Tot.Var.}_f(\bar{t}-) + C\,|f(\rho_l) - f(\rho_r)| \\
&= \text{Tot.Var.}_f(\bar{t}-) - |f(\rho_l) - f(\rho_r)| + (C+1)\,|f(\rho_l) - f(\rho_r)| \\
&\leq \max\{C+1, 1\}\,[\text{Tot.Var.}_f(\bar{t}-) - |f(\rho_l) - f(\rho_r)|] \\
&\quad + \max\{C+1, 1\}\,|f(\rho_l) - f(\rho_r)| \\
&= (C+1)\text{Tot.Var.}_f(\bar{t}-),
\end{aligned}$$

and this concludes the proof. □

Lemma 5.4 *Assume that a wave (ρ_l, ρ_r) interacts with J at a time $\bar{t} > 0$. Then*

$$\Gamma(\bar{t}+) \le \Gamma(\bar{t}-) + (C+2)\left|f(\rho_l) - f(\rho_r)\right|, \tag{37}$$

where C is given by property (P2).

PROOF. The variation of Γ at $\bar{t}$ is the sum of the variation of the fluxes for the incoming roads. Therefore

$$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) \le \text{Tot.Var.}_f(\bar{t}+) - \text{Tot.Var.}_f(\bar{t}-) + 2\left|f(\rho_l) - f(\rho_r)\right|.$$

Hence, by (P2), it is bounded by $(C+2)\left|f(\rho_l) - f(\rho_r)\right|$. □

The next lemma gives a bound for the positive total variation of Γ.

Lemma 5.5 *We have*

$$\mathit{Tot.Var.}^{+}\,\Gamma(\cdot) \le (C+2)\,\mathit{Tot.Var.}_f(0+), \tag{38}$$

where C is given by property (P2) and $\mathit{Tot.Var.}^{+}\,\Gamma(\cdot)$ denotes the positive total variation of Γ.

PROOF. By Lemma 5.2, an increment of the functional Γ can happen only when a wave with increasing flux interacts with J. Moreover a wave, generated at J, can come back at J only with a decreasing flux. Indeed, consider the case of an incoming road, the other one being completely symmetric. Assume that a wave with increasing flux (ρ_l, ρ_r) interacts with J. Since $f(\rho_l) > f(\rho_r)$ and the velocity of the wave is positive, then we deduce that $\rho_l > \rho_r$. By contradiction, if $\rho_l > \sigma$, then clearly $\rho_r \in [0, \tau(\rho_l)[$ and so (ρ_l, ρ_r) is a rarefaction wave, whose velocity is not positive. Hence $\rho_l \le \sigma$ and ρ_r is a bad datum. By [23, Lemma 4.3.6] (see the Appendix), the wave (ρ_l, ρ_r) is not a wave coming back to J. Thus we conclude, by using Lemma 5.4. □

Lemma 5.6 *For C given by property (P2), we have*

$$\mathit{Tot.Var.}\,\Gamma(\cdot) \le 2(C+2)\,\mathit{Tot.Var.}_f(0+) + nf(\sigma). \tag{39}$$

PROOF. It is a direct consequence of Lemma 5.5 and of the bound (34). □

Lemma 5.7 *For every $t > 0$ we have*

$$\mathit{Tot.Var.}_f(t) \le C_1\,\mathit{Tot.Var.}_f(0+) + Cnf(\sigma), \tag{40}$$

where $C_1 = 1 + 2C(C+2)$ and C is given by property (P2).

PROOF. Notice that the functional Tot.Var.$_f$ can increase only when a wave interacts with J and, by property (P2) of the Riemann solver $\mathcal{RS}$, produces a variation of Γ. If we denote with $g(t)$ the function Tot.Var.$_f(t)$, then the positive variation Tot.Var.$^+ g(\cdot)$ of g is bounded by $C \cdot$ Tot.Var. $\Gamma(\cdot)$. Thus, by Lemma 5.6,

$$\text{Tot.Var.}^+ g(\cdot) \leq 2C(C+2)\text{Tot.Var.}_f(0+) + Cnf(\sigma)$$

and so, for every $t > 0$,

$$\begin{aligned} \text{Tot.Var.}_f(t) &\leq \text{Tot.Var.}_f(0+) + 2C(C+2)\text{Tot.Var.}_f(0+) + Cnf(\sigma) \\ &= C_1\text{Tot.Var.}_f(0+) + Cnf(\sigma). \end{aligned}$$

The proof is finished. □

We conclude the section with the proof of the main result.

PROOF OF THEOREM 5.1. By Lemma 5.7, we deduce that there exists a constant $M > 0$, depending on the total variation of the flux of the initial datum, such that

$$\text{Tot.Var.}_f(\cdot) \leq M.$$

For every $l \in \{1, \ldots, n+m\}$ and every $\nu \in \mathbb{N}$, using the concept of generalized characteristic introduced by Dafermos [16], we construct a curve $Y_{l,\nu}$ bounding the region of influence of waves generated by the junction J on the approximate solution $\rho_{l,\nu}$. More precisely, we follow the generalized characteristic emanating from 0 at time 0, sticking to the boundary of I_l each time $Y_{l,\nu}$ is at 0 and the characteristic speed is positive (resp. negative) if I_l is an incoming (resp. outgoing) road. The curve $Y_{l,\nu} : [0, +\infty[\to I_l$ then satisfies

1. $Y_{l,\nu}(0) = 0$;
2. in $D_1^{l,\nu} = \{(t,x) \in [0,+\infty[\times I_l : |x| \geq Y_{l,\nu}(t)\}$, the function $\rho_{l,\nu}$ depends only on the initial condition;
3. in $D_2^{l,\nu} = [0, +\infty[\times I_l \setminus D_1^{l,\nu}$ the function $\rho_{l,\nu}$ depends also on the data from other roads and on the Riemann solver $\mathcal{RS}$.

By uniform Lipschitz continuity, possibly by passing to a subsequence, the curves $Y_{l,\nu}$ converge uniformly as $\nu \to +\infty$ to some Lipschitz continuous limit curves. Thus for every $l \in \{1, \ldots, n+m\}$, there exist two sets $D_1^l, D_2^l \subseteq [0, +\infty[\times I_l$, which are "limits" of the regions $D_1^{l,\nu}$, $D_2^{l,\nu}$, in the sense that

$\text{meas}(D_1^l \Delta D_1^{l,\nu}) \to 0$ and $\text{meas}(D_2^l \Delta D_2^{l,\nu}) \to 0$, where Δ indicates the set-theoretic symmetic difference.
For every $l \in \{1, \dots, n+m\}$, $\rho_{l,\nu}$ converges to a limit function ρ_l in L^1_{loc} on D_1^l by the theory of conservation laws on a real line, see [9].
Now recall that, for every $l \in \{1, \dots, n+m\}$ and $\nu \in \mathbb{N}$, $\rho_{l,\nu} \in L^\infty$. Therefore, possibly up to a subsequence, on D_2^l the sequence $\rho_{l,\nu}$ weakly converges to a limit function ρ_l in L^1 and $f(\rho_{l,\nu})$ strongly converges to $\bar{f}_l$ in L^1 for some $\bar{f}_l$. By [23, Lemma 4.3.6] (see the Appendix), for every $\bar{t}$ the set $D_2^{l,\nu} \cap \{(t,x) : t = \bar{t}\}$ contains at most one big shock. This permits to conclude that $\rho_{l,\nu}$ strongly converges to ρ_l (being f invertible possibly subdividing futherly $D_2^{l,\nu}$). Finally, the vector $(\rho_1(t,x), \dots, \rho_{n+m}(t,x))$ is a weak solution at J satisfying 1 and 2 of Theorem 5.1. □

Remark 8 *In the case of Riemann solver $\mathcal{RS}_2$, when a wave interacts with J at time $\bar{t}$, the total variation of the flux does not change, i.e.*

$$Tot.Var._f(\bar{t}-) = Tot.Var._f(\bar{t}+).$$

For a detailed proof of this fact, see Lemma 4.6. Therefore the constant M in the proof of Theorem 5.1 can be chosen equal to $Tot.Var._f(0+)$.

6 Dependence of solutions on initial data

It is known that the Lipschitz continuous dependence of the solution to the Cauchy problem (29) with respect to the initial datum in general does not hold in the case of Riemann solver $\mathcal{RS}_1$. More precisely in [12, 23] there is a counterexample to the Lipschitz continuous dependence property in the case of a junction with two incoming and two outgoing roads.

On the other side, the Lipschitz continuous dependence of the solution to (29) with respect to the initial datum was proved in the case of Riemann Solver $\mathcal{RS}_2$ and simple junctions in [18]; see also [23]. In this section we want to prove that the property holds for every type of junctions.
Let us introduce the concept of Finsler manifold.

Definition 6.1 *Consider a differentiable manifold M and, for every $x \in M$, a norm $\|\cdot\|_x$ on the tangent space T_xM. The manifold M is said a Finsler manifold if*

1. *$\|\cdot\|_x$ depends in a continuous way on x;*

2. for every $x \in M$ and $v \in T_xM$ the Hessian of the function

$$\begin{array}{rcl} L_x : T_xM & \longrightarrow & \mathbb{R} \\ w & \longmapsto & \|w\|_x^2 \end{array}$$

is positive definite at v.

Given a Finsler manifold M, a metric d is naturally defined by:

$$d(x,y) = \inf_{\Omega(x,y)} \int_0^1 \|\dot{\gamma}(\theta)\| \, d\theta$$

where $\Omega(x,y)$ is the set of smooth curves $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Our main idea is to put a Finsler type structure on $L^1(\mathbb{R})$, which measures the norm of generalized tangent vectors and is not defined on the whole space, thus not ensuring the second property of Definition 6.1. To do this we first focus on piecewise constant functions and define "generalized" tangent vectors in terms of shift of discontinuities. Still we can define a distance among piecewise constant function, which happens to coincide with the usual L^1 metric and thus can be naturally extended to the whole L^1. The difference is in the differential structure at the base of this new metric, which will permit to prove the Lipschitz continuous dependence.

Consider a curve $\gamma : [0,1] \to L^1$ taking values on the set of piecewise constant functions with N discontinuities, indicating by $x_1(\theta) < x_2(\theta) < \cdots < x_N(\theta)$ the discontinuity points of $\gamma(\theta)$. Then γ admits as tangent vector $(v,\xi)(\theta) \in L^1 \times \mathbb{R}^N$ if the following holds:

$$L^1 \ni v(\theta,x) = \lim_{h\to 0} \frac{\gamma(\theta+h,x) - \gamma(\theta,x)}{h}, \qquad \text{for a.e. } x,$$

$$\xi_i(\theta) \doteq \lim_{h\to 0} \frac{x_i(\theta+h) - x_i(\theta)}{h}, \qquad i = 1,...,N.$$

In this case we write $\dot{\gamma}(\theta) = (v,\xi)(\theta)$. Notice that γ is not differentiable according to the usual differential structure of L^1, since the L^1-limit of $\gamma(\theta + h) - \gamma(\theta)/h$ does not exist (indeed such ratio converges to a finite sum of Dirac deltas).
The norm of $(v,\xi)(\theta)$ is defined by:

$$\|(v,\xi)(\theta)\| = \|v(\theta)\|_{L^1} + \sum_{i=1}^{N} |\xi_i(\theta)||\gamma(\theta,x_i+) - \gamma(\theta,x_i-)|.$$

The norm of (v,ξ) measures precisely the infinitesimal L^1 displacement of γ. Then, for every couple of piecewise constant functions $u, u' \in L^1$ we can define the distance:

$$d(u,u') = \inf_{\Omega(u,u')} \int_0^1 \|\dot\gamma(\theta)\| \, d\theta$$

where $\Omega(u,u')$ is the set of curves $\gamma : [0,1] \to L^1$ admitting piecewise smooth tangent vectors (thus having a piecewise constant number of discontinuities), such that $\gamma(0) = u$ and $\gamma(1) = u'$. We easily get that d coincides with the usual L^1 distance (since we defined suitably the norm of tangent vectors). Then d can be extended to the whole L^1 using the usual L^1 distance, namely we can set:

$$d(u,u') = \inf\{\|u - w\|_{L^1} + d(w,w') + \|w' - u'\|_{L^1} : w, w' \text{ piecewise constant}\}.$$

Moreover d can be recovered just using curves with tangent vectors having a zero L^1 component. More precisely:

Lemma 6.1 *Given u, $u' \in L^1$ piecewise constant, let us indicate by $\widetilde\Omega(u,u')$ the set of curves $\gamma : [0,1] \to L^1$, $\gamma(0) = u$, $\gamma(1) = u'$, admitting piecewise smooth tangent vectors (v,ξ) such that $v \equiv 0$. Then it holds:*

$$\inf_{\widetilde\Omega(u,u')} \int_0^1 \|\dot\gamma(\theta)\| \, d\theta = \inf_{\Omega(u,u')} \int_0^1 \|\dot\gamma(\theta)\| \, d\theta = d(u,u') = \|u - u'\|_{L^1}.$$

Proof. Consider the curve defined for $t \in]0,1[$ by:

$$\gamma(t) = u(t)\chi_{]-\infty, \tan\left(\pi t - \frac{\pi}{2}\right)]} + u'(t)\chi_{]\tan\left(\pi t - \frac{\pi}{2}\right), \infty[},$$

where χ is the indicator function, and setting, by continuity, $\gamma(0) = u$ and $\gamma(1) = u'$. Then clearly γ admits a piecewise smooth tangent vector (v,ξ) with $v \equiv 0$. Indeed, for every t such that $x(t) = \tan\left(\pi t - \frac{\pi}{2}\right)$ is neither a discontinuity point of u nor of u' we get:

$$\dot\gamma(t) = \left(0, \pi \left[1 + \tan^2\left(\pi t - \frac{\pi}{2}\right)\right] [u'(x(t)) - u(x(t))]\right).$$

Moreover, the second component of $\dot\gamma$ spans exactly the area contained between the graphs of u and u' so that:

$$\int_0^1 \|\dot\gamma(\theta)\| \, d\theta = \|u - u'\|_{L^1},$$

which gives the conclusion. □

Remark 9 *The tecnique of generalized tangent vectors was used in [10] for systems. In that case one has to introduce weights in the definition of the norm of a tangent vector. Therefore d happens to be equivalent but not equal to the L^1 metric. Moreover Lemma 6.1 does no more hold true.*

Now the main idea to prove Lipschitz continuous dependence is the following. We consider the same Finsler structure on the set $L^1(\Pi_l I_l)$. Given two initial data $\rho(0)$ and $\rho'(0)$, we focus on wave-front tracking approximate solutions $\rho_\nu(t)$, $\rho'_\nu(t)$ and for every $\gamma_0 \in \Omega(\rho(0), \rho'(0))$ define γ_t to be the evolution of γ_0 at time t. Assume we can prove that γ_t admits a tangent vector $(v,\xi)_t$ such that:

$$\|(v,\xi)_t(\theta)\| \leq \|(v,\xi)_0(\theta)\|. \tag{41}$$

Then, denoting by Ω_t the curves obtained by evolutions of $\gamma_0 \in \Omega(\rho(0), \rho'(0))$, we get:

$$d(\rho_\nu(t), \rho'_\nu(t)) = \inf_{\Omega(\rho_\nu(t), \rho'_\nu(t))} \int_0^1 \|\dot{\gamma}(\theta)\|\, d\theta \leq$$

$$\inf_{\Omega_t} \int_0^1 \|\dot{\gamma}(\theta)\|\, d\theta = \inf_{\Omega_t} \int_0^1 \|(v,\xi)_t(\theta)\|\, d\theta \leq$$

$$\inf_{\Omega(\rho(0), \rho'(0))} \int_0^1 \|(v,\xi)_0(\theta)\|\, d\theta = d(\rho_\nu(0), \rho'_\nu(0)).$$

Passing to the limit in ν and recalling that d coincides with the usual L^1 metric, we conclude the Lipschitz continuous dependence on initial data.

Let us now pass to estimates on the shift of waves along wave-front tracking approximate solutions. We start with a definition.

Definition 6.2 *Fix $\xi \in \mathbb{R}$ and a wave (ρ_l, ρ_r) of an ε-approximate wave-front tracking solution to (29). We say that ξ forms a shift for the wave (ρ_l, ρ_r) if we consider the same ε-approximate wave-front tracking solution, except for the position of the wave (ρ_l, ρ_r), which is translated by the quantity ξ in the x-direction.*

The proof of the continuous dependence is based on the following general lemma.

Lemma 6.2 *Fix an ε-approximate wave-front tracking solution to (29) $\bar{\rho}_\varepsilon$ with respect to a Riemann solver $\mathcal{RS}$, satisfying the consistency condition. Assume that a wave $(\hat{\rho}_k^-, \tilde{\rho}_k^-)$ in a road I_k ($k \in \{1, \ldots, n+m\}$) interacts with J at time $\bar{t} > 0$ producing waves $(\hat{\rho}_l^+, \tilde{\rho}_l^+)$ in (possibly) all the roads of the* *junction J. If the interacting wave in I_k is shifted by ξ_k^-, then all the waves produced at J at time $\bar{t}$ are shifted by ξ_l^+ ($l \in \{1, \ldots, n+m\}$), which satisfy the relations*

$$\left| \xi_k^- \frac{\hat{\rho}_k^- - \tilde{\rho}_k^-}{f(\hat{\rho}_k^-) - f(\tilde{\rho}_k^-)} \right| = \left| \xi_l^+ \frac{\hat{\rho}_l^+ - \tilde{\rho}_l^+}{f(\hat{\rho}_l^+) - f(\tilde{\rho}_l^+)} \right| \tag{42}$$

for every $l \in \{1, \ldots, n+m\}$.

Proof. Note that, applying the shift ξ_k^-, the interaction of the wave $(\hat{\rho}_k^-, \tilde{\rho}_k^-)$ with J is shifted in time by

$$\xi_k^- \frac{\hat{\rho}_k^- - \tilde{\rho}_k^-}{f(\hat{\rho}_k^-) - f(\tilde{\rho}_k^-)} .$$

The shift in time of the waves generated by this interaction must be the same and so the proof easily follows. □

Theorem 6.1 *Fix $\boldsymbol{\theta} \in \Theta$ and consider the Cauchy problem (29) with the Riemann solver $\mathcal{RS}_2$. There exists a unique $(\rho_1(t,x), \ldots, \rho_{n+m}(t,x))$, weak solution at J, such that*

1. *for every $l \in \{1, \ldots, n+m\}$, $\rho_l(0,x) = \rho_{0,l}(x)$ for a.e. $x \in I_l$;*
2. *for a.e. $t > 0$,*

$$\mathcal{RS}_2(\rho_1(t,0), \ldots, \rho_{n+m}(t,0)) = (\rho_1(t,0), \ldots, \rho_{n+m}(t,0)) . \tag{43}$$

Moreover the solution depends in a Lipschitz continuous way on the initial datum with respect to the L^1-topology.

Proof. As explained above, we can restrict to estimate the L^1-distance among wave-front tracking solutions. For this, it is enough to show that

$$\|(v,\xi)_t(\theta)\| \leq \|(v,\xi)_0(\theta)\|$$

for every $t > 0$ and $\theta \in [0,1]$. We prove the latter estimating the evolution of the tangent vector norm at each time. Moreover, by Lemma 6.1, we can restrict the study to the evolution of shifts.
Fix a time $\bar{t} > 0$ and, without loss of generality, treat the following cases:

a) no interaction of waves takes place in any road at $\bar{t}$ and no wave interacts with J;

b) two waves interact at $\bar{t}$ on a road and no other interaction takes place;

c) a wave interacts with J at $\bar{t}$ and no other interaction takes place.

In case a) the shifts are constant, while in case b) the norms are decreasing by Lemma 2.7.2 of [23] (see the Appendix) .
Assume now that a wave $(\hat{\rho}_{\bar{l}}^-, \tilde{\rho}_{\bar{l}}^-)$ interacts with J at time $\bar{t}$ from the road $I_{\bar{l}}$. Using Lemma 4.6 and Lemma 6.2, we deduce

$$\begin{aligned}\|(v,\xi)(\bar{t}+)\| - \|(v,\xi)(\bar{t}-)\| &= \sum_{l=1}^{n+m} \left|\xi_l^+\right| \left|\hat{\rho}_l^+ - \tilde{\rho}_l^+\right| - \left|\xi_{\bar{l}}^-\right| \left|\hat{\rho}_{\bar{l}}^- - \tilde{\rho}_{\bar{l}}^-\right| \\ &= \left[\sum_{l=1}^{n+m} \left|\frac{f(\hat{\rho}_l^+) - f(\tilde{\rho}_l^+)}{f(\hat{\rho}_{\bar{l}}^-) - f(\tilde{\rho}_{\bar{l}}^-)}\right| - 1\right] \left|\xi_{\bar{l}}^-\right| \left|\hat{\rho}_{\bar{l}}^- - \tilde{\rho}_{\bar{l}}^-\right| \\ &= 0\,,\end{aligned}$$

where $(\hat{\rho}_l^+, \tilde{\rho}_l^+)$ is the wave produced in the road I_l after the interaction. This estimate permits to conclude. □

A Technical lemmas

In this section we report the statements of Lemmas 2.7.2, 4.3.6 and 4.3.7 of [23], for readers' convenience.

Lemma 2.7.2 of [23]. Consider two waves, with speeds λ_1 and λ_2 respectively, that interact together at $\bar{t}$ producing a wave with speed λ_3. If the first wave is shifted by ξ_1 and the second wave by ξ_2, then the shift of the resulting wave is given by

$$\xi_3 = \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_2}\xi_1 + \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}\xi_2. \tag{44}$$

Moreover we have that

$$\Delta\rho_3\,\xi_3 = \Delta\rho_1\,\xi_1 + \Delta\rho_2\,\xi_2, \tag{45}$$

where $\Delta\rho_i$ are the signed strengths of the corresponding waves.

Lemma 4.3.6 of [23]. If a road I_l of a junction J has a good datum, then it remains good after interactions with J of waves coming from other roads. Moreover, no big shock can be produced in this way. If a road I_l has a bad datum, then after any interaction with J of waves coming from other roads, either the datum of I_l is unchanged or a big shock is produced (and the new datum is good).

Lemma 4.3.7 of [23]. If a wave produced from a junction J on an incoming road I_i comes back to J, interacting only with waves produced by J, then the wave connects a bad left datum to a right good datum. The converse is true for outgoing roads.

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