Coupling of LWR and phase transition models at boundary

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Abstract

This paper deals with coupling conditions between the classical Lighthill-Whitham-Richards (LWR) model and a phase transition (PT) model, introduced in [3]. We propose two different definitions of solution at the interface between the two models. The first one corresponds to maximize the flux passing through the interface, while the second one imposes an additional constraint on the flux. We prove existence of solutions to the Cauchy problem with arbitrary initial data of bounded variation, by means of the wave-front tracking technique.

Key Words: LWR model, phase transition model, conservation laws, coupling conditions, Cauchy Problem, wave-front tracking technique.

AMS Subject Classifications: 90B20, 35L65.

1 Introduction

In recent years a significant effort was devoted by applied mathematician to the modeling of vehicular traffic via partial differential equations, thus dealing with the macroscopic scale of the problem. Such effort enriched the already wide spectrum of available mathematical models, many of which were proposed by the transportation engineering community and focused on the microscopic scale. The landscape completed by kinetic (mesoscopic) models and multiscale ones.

Therefore researchers and practitioners face now an unprecedented opportunity in model choice. In turn this poses the problem of identifying the "best" model. It is common opinion (and probably common sense) that a unique best model does not exist. It is rather true that different scales and mathematical frameworks present different characteristics, thus the best modelling approach is obtained by combining more than one model. This may mean to use a multiscale setting. Another typical situation is that of a large complex network, with a small portion of it requiring a fine modeling and the remaining part for which a coarse model is sufficient. This leads to the necessity of combining different models using coupling conditions at interfaces.

Let us start revising briefly some of the most well known models. At microscopic level, one of the most studied approach is the so called Follow the Leader, where each driver adjusts its velocity to the one of the vehicle in front. More precisely, the velocity change is proportional to difference in velocities and inversely proportional to vehicles distance. The fluid dynamic approach at macroscopic level was, in turn, initiated by the seminal work of Lighthill-Whitham and Richards [19, 22], nowadays known as the LWR model. The LWR model well describes the evolution of free traffic, but it is not accurate when the traffic is congested. Hence second order models, i.e. system with two equations, were introduced. Among these we mention the Payne-Whitham [21, 23] model, which was invalidated by Daganzo [11] in 1995. Aw and Rascle in [1] then proposed

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the conservation of a "modified" momentum. The same model was independently suggested by Zhang in [25], thus we refer to it as the Aw–Rascle–Zhang (or briefly ARZ) model. In 2002, Colombo proposed a second order model with phase transitions [5], in order to describe the different behaviors of traffic in free and congested regimes. It is based on two phases: the free and the congested one. In free flow the model is a classical LWR model, whereas in the congested flow it is a system of two equations: the first one is the conservation of the number of vehicles, while the second one describes the evolution of a linearized momentum. Various modifications of this model were considered, for instance combining LWR and ARZ model [14] and the more recent refinement [3]; see also [9].

The relations among microscopic and (fluid dynamic) macroscopic were studied in few papers. In [2], the authors showed how solutions to the Follow the Leader model, with appropriate rescaling, in the limit tend to solutions of the ARZ model. In [20] an hybrid model is built based on results from [2]. Coupling at boundaries has been investigated first only under a numerical point of view (see, for instance, [15, 16]). Then in [18] a coupling at boundaries of the Follow the Leader and ARZ model was proposed.

The simulation of large networks and fitting of data from mobile sensors is mostly performed using the classical LWR model or modifications of it; see [24, 10]. The combination of different scales thus requires the definition of coupling conditions at boundaries between the Follow the Leader model and the LWR ones. The main obstacle along this path is the fact that the Follow the Leader model is naturally coupled with second order ones (that means two equations) as explained above. To overcome this difficulty, in this paper we define coupling conditions for the LWR and the phase transition (briefly PT) model of [3]. This opens the way towards the coupling of microscopic models of Follow the Leader type with first order models such as LWR, via an intermediate coupling with second order models.

In the paper we assume that there is a fixed interface, located at $x = 0$, between the LWR and the PT model. We give coupling conditions between these two models, based on the conservation of the number of vehicle passing through the interface. As for the case of junctions in road networks, the constraint about the conservation of the number of cars is not sufficient to isolate a unique solution. Therefore we also impose an additional rule based on the maximization of the flux through the interface, possibly with a flux constraint, producing two different kinds of solutions at $x = 0$. Such types of flux constraints are used, for example, in the modeling of toll gates, see [7, 8].

We also consider the Cauchy problem for our system and we prove existence of solutions by means of the wave-front tracking technique. The key estimate to obtain compactness for the sequence of approximated solutions is a uniform bound of the total variation of the flux (for the LWR model) and of a functional measuring the strength of waves (for the PT model). We remark that these functionals are not decreasing along wave-front tracking approximate solutions; this is due to some wave interactions. However a detailed analysis of all the possible interactions permits to deduce, for such functionals, an exponential type bound, uniform with respect to the time. Moreover, we also prove that the number of waves and interactions is finite for wave-front tracking approximate solutions.

The paper is organized as follows. In Section 2, we recall the basic definitions of the Lighthill-Whitham-Richards model, whereas in Section 3 the phase transition one, introduced in [3], is presented. Sections 4 and 5 deal with coupling conditions and with the analysis for the Cauchy problems respectively for LWR-PT model and for PT-LWT model. Finally an Appendix, containing some technical proofs and the list of all the interactions involving a phase transition wave, concludes the paper.

2 Description of the LWR model

The LWR model is described by the scalar hyperbolic conservation law

$$\partial_t \rho + \partial_x(\rho \tilde{v}) = 0 \tag{1}$$

where $\rho \in [0, R]$ denotes the density of cars in a road and $\tilde{v}$ is the average velocity. We assume that $\tilde{v}$ is a given decreasing function of the density, in such a way the flux $f(\rho) = \rho\tilde{v}(\rho)$ is a smooth concave function satisfying

1. $f(0) = f(R) = 0$;

2. there exists a unique point $\tilde{\sigma} \in (0, R)$ such that $f'(\tilde{\sigma}) = 0$.

For simplicity, in the paper we consider the special case

$$\begin{array}{rcl} \tilde{v} : [0, R] & \longrightarrow & \mathbb{R} \\ \rho & \longmapsto & V_{max}\left(1 - \frac{\rho}{R}\right) \end{array} \tag{2}$$

PSfrag replacements

ρ

$\tilde{v}$

f

0

σ

R

V_{max}

Figure 1: The velocity function and the fundamental diagram for the LWR model.

where $V_{max} > 0$ is the maximum speed. In this case we have $f(\rho)$ equal to $V_{max}\rho\left(1 - \frac{\rho}{R}\right)$ and $\tilde{\sigma} = \frac{R}{2}$; see Figure 1.

For $\rho \in [0, R]$, define the following sets

$$\mathcal{T}_l^1(\rho) = \begin{cases} \{\rho\} \cup]R - \rho, R], & \text{if } \rho < \frac{R}{2}, \\ \left[\frac{R}{2}, R\right], & \text{if } \rho \geq \frac{R}{2}, \end{cases} \tag{3}$$

and

$$\mathcal{T}_r^1(\rho) = \begin{cases} \left[0, \frac{R}{2}\right], & \text{if } \rho \leq \frac{R}{2}, \\ \{\rho\} \cup [0, R - \rho[, & \text{if } \rho > \frac{R}{2}. \end{cases} \tag{4}$$

We have the following characterization of the sets $\mathcal{T}_l^1(\rho)$ and $\mathcal{T}_r^1(\rho)$; for a proof see [12].

Proposition 2.1 *Fix a density $\bar{\rho} \in [0, R]$.*

1. *For every $\tilde{\rho} \in \mathcal{T}_l^1(\bar{\rho})$, the Riemann problem*

$$\begin{cases} \partial_t \rho + \partial_x(\rho\tilde{v}) = 0, & \textit{if } x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = \bar{\rho}, & \textit{if } x \leq 0, \\ \rho(0, x) = \tilde{\rho}, & \textit{if } x > 0, \end{cases}$$

is solved with waves with non positive speed.

2. *For every $\tilde{\rho} \in \mathcal{T}_r^1(\bar{\rho})$, the Riemann problem*

$$\begin{cases} \partial_t \rho + \partial_x(\rho\tilde{v}) = 0, & \textit{if } x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = \tilde{\rho}, & \textit{if } x < 0, \\ \rho(0, x) = \bar{\rho}, & \textit{if } x \geq 0, \end{cases}$$

is solved with waves with non negative speed.

3 Description of the phase transition model

We describe the phase transition model, introduced in [3], with the Newell-Daganzo velocity function for congestion. The model consists in two phases: the free and the congested. These phases are described by the sets

$$\begin{cases} \Omega_f = \left\{(\rho, q) \in [0, R] \times [0, +\infty[:\ q = \frac{R(\rho - \sigma)}{\sigma(R - \rho)},\ 0 \leq \rho \leq \sigma_+\right\} \\ \Omega_c = \left\{(\rho, q) \in [0, R] \times [0, +\infty[:\ v_c(\rho, q) \leq V,\ \frac{q^-}{R} \leq \frac{q}{\rho} \leq \frac{q^+}{R}\right\} \end{cases} \tag{5}$$

where $V > 0$ is the velocity in the free phase, $R > 0$ is the maximal density, $-1 < q^- < 0 < q^+ < 1$, $\frac{R}{5} < \sigma < R$ and $v_c : [\sigma_-, R] \times [q^-, q^+] \to [0, +\infty[$, defined by

$$v_c(\rho, q) = v_c^{eq}(\rho)(1 + q) = \frac{V\sigma}{R - \sigma}\left(\frac{R}{\rho} - 1\right)(1 + q), \tag{6}$$

is the velocity in the congested phase; see Figure 2. Here the constants $\sigma_- > 0$ and $\sigma_+ > 0$ are defined respectively by the ρ-component of the solutions in Ω_c to the systems

$$\begin{cases} v_c(\rho, q) = V, \\ \frac{q}{\rho} = \frac{q^-}{R}, \end{cases} \qquad \text{and} \qquad \begin{cases} v_c(\rho, q) = V, \\ \frac{q}{\rho} = \frac{q^+}{R}. \end{cases}$$

PSfrag replacements

R

ρ

0

-1

q

ρv

q^-

q^+

σ

σ_+

σ_-

Figure 2: The fundamental diagram for the phase transition model in (ρ, q) and $(\rho, \rho v_c)$ coordinates.

The model in the free phase Ω_f reads

$$\partial_t \rho + \partial_x(\rho V) = \partial_t \rho + V \partial_x(\rho) = 0, \tag{7}$$

while in the congested phase Ω_c reads

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho, q)) = 0. \end{cases} \tag{8}$$

The eigenvalues for (8) are

$$\lambda_1(\rho, q) = (1 + 2q) v_c^{eq}(\rho) + \rho(1 + q)(v_c^{eq})'(\rho), \quad \lambda_2(\rho, q) = (1 + q) v_c^{eq}(\rho), \tag{9}$$

while the respectively eigenvectors are

$$r_1 = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad r_2 = \begin{pmatrix} v_c^{eq}(\rho) \\ -(1+q)(v_c^{eq})'(\rho) \end{pmatrix}. \tag{10}$$

Moreover the first characteristic speed is genuinely nonlinear if $q \neq 0$ and it is linearly degenerate if $q = 0$. Instead the second characteristic speed is linearly degenerate. A deeper analysis of (8) is contained in [3]. We denote with $L_1(\rho; \rho_0, q_0)$ and $L_2(\rho; \rho_0, q_0)$ respectively the Lax curves of the first and second family for system (8); see Figure 3. We recall that $(\rho, L_1(\rho; \rho_0, q_0))$ are lines in the (ρ, q) plane passing through the origin and $(\rho, L_2(\rho; \rho_0, q_0))$ are lines in the $(\rho, \rho v)$ plane passing through the origin. Let us introduce the functions

$$\psi_1 : \Omega_c \to \Omega_c, \qquad \psi_2^- : \Omega_c \to \Omega_c, \qquad \psi_2^+ : \Omega_c \to \Omega_c, \tag{11}$$

such that, for every $(\rho_0, q_0) \in \Omega_c$, $\psi_1(\rho_0, q_0)$, $\psi_2^-(\rho_0, q_0)$ and $\psi_2^+(\rho_0, q_0)$ are defined respectively by the solutions in Ω_c to the systems

$$\begin{cases} \frac{q}{\rho} = \frac{q_0}{\rho_0}, \\ v_c(\rho, q) = V, \end{cases} \qquad \begin{cases} \frac{q}{\rho} = \frac{q^-}{R}, \\ v_c(\rho, q) = v_c(\rho_0, q_0), \end{cases} \qquad \begin{cases} \frac{q}{\rho} = \frac{q^+}{R}, \\ v_c(\rho, q) = v_c(\rho_0, q_0). \end{cases}$$

Moreover, define the flux function

$$\begin{array}{rcl} \varphi : \Omega_f \cup \Omega_c & \longrightarrow & \mathbb{R} \\ (\rho, q) & \longmapsto & \begin{cases} \rho V, & \text{if } (\rho, q) \in \Omega_f, \\ \rho v_c(\rho, q), & \text{if } (\rho, q) \in \Omega_c. \end{cases} \end{array}$$

Definition 3.1 *Let I be a real interval. We say that a function $(\rho, q) : I \to \Omega_f \cup \Omega_c$ satisfies the assumption **(H-1)** in I if there exists $0 < \bar{v} < V$ such that, for a.e. $x \in I$, $(\rho, q)(x) \in \Omega_f$ or $(\rho, q)(x) \in \Omega_c$ and $v_c((\rho, q)(x)) \geq \bar{v}$.*

Definition 3.2 *We say that the phase transition model, described by (5), (6), (7), (8), satisfies the assumption **(H-2)** if, for every (ρ_A, q_A), $(\rho_B, q_B) \in \Omega_c$ and for every (ρ'_A, q'_A), $(\rho'_B, q'_B) \in \Omega_c$ such that $\rho_A < \rho'_A$, $\rho_B < \rho'_B$, $\frac{q_A}{\rho_A} = \frac{q_B}{\rho_B}$, $\frac{q'_A}{\rho'_A} = \frac{q'_B}{\rho'_B}$, $v_c(\rho_A, q_A) = v_c(\rho'_A, q'_A)$ and $v_c(\rho_B, q_B) = v_c(\rho'_B, q'_B)$, we have*

$$|\varphi(\rho_A, q_A) - \varphi(\rho_B, q_B)| \leq |\varphi(\rho'_A, q'_A) - \varphi(\rho'_B, q'_B)|.$$

PSfrag replacements

R

ρ

0

-1

q

ρv

L_1

L_2

(ρ_0, q_0)

$(\rho_0, \rho_0 v_c(\rho_0, q_0))$

ψ_1

ψ_2^-

ψ_2^+

Figure 3: Shape of Lax curves in (ρ, q) and $(\rho, \rho v_c(\rho, q))$ planes.

Remark 1 *There are suitable choices of the parameters such that the phase transition model satisfies the assumption **(H-2)**. Indeed, let us consider $R = 1$, $V = 1$, $\sigma = 1/2$, $q^+ = 1/2$, $q^- = -1/2$. We easily deduce that $\sigma_- = \frac{5-\sqrt{17}}{2}$. Take for simplicity two points $(\rho_A, q_A) \in \Omega_c$ and $(\rho_B, q_B) \in \Omega_c$ such that $\psi_2^-(\rho_A, q_A) = (\rho_A, q_A)$, $\psi_2^-(\rho_B, q_B) = (\rho_B, q_B)$ and $\sigma_- \le \rho_A < \rho_B < R$. Let (ρ'_A, q'_A), $(\rho'_B, q'_B) \in \Omega_c$ be such that $\frac{q'_A}{\rho'_A} = \frac{q'_B}{\rho'_B}$, $v_c(\rho_A, q_A) = v_c(\rho'_A, q'_A)$ and $v_c(\rho_B, q_B) = v_c(\rho'_B, q'_B)$. In this situation $\varphi(\rho'_B, q'_B) > \varphi(\rho'_A, q'_A)$. Note that the points (ρ'_A, q'_A) and (ρ'_B, q'_B) depend on (ρ_A, q_A) and (ρ_B, q_B). Define*

$$h_1(\rho'_A, q'_A) = \lim_{\rho_B \to \rho_A} \frac{\varphi(\rho'_B, q'_B) - \varphi(\rho'_A, q'_A)}{\rho_B - \rho_A}$$

and

$$h_2(\rho'_A, q'_A) = \frac{d}{d\rho'_A} h_1(\rho'_A, q'_A).$$

The exact expression for $h_2(\rho'_A, q'_A)$, found with the symbolic calculator Maxima, is

$$\begin{aligned}
&\Big[{\rho'_A}^3 \left(2\,{\rho_A}^4 - 4\,{\rho_A}^3 + 8\,\rho_A - 8\right) \\
&+ {\rho'_A}^2 \left(-{\rho_A}^4 - {\rho_A}^3 + 2\,\rho_A + 4\right) + \rho'_A \left(2\,{\rho_A}^3 - 4\,\rho_A\right)\Big] \\
&\cdot \frac{\left(2\,\rho'_A \left(2\,{\rho_A}^4 - 2\,{\rho_A}^3 + 4\,{\rho_A}^2\right) - 8\,{\rho_A}^3\right)}{\left({\rho'_A}^2 \left(2\,{\rho_A}^4 - 2\,{\rho_A}^3 + 4\,{\rho_A}^2\right) - 8\,\rho'_A\,{\rho_A}^3 + 4\,{\rho_A}^3\right)^2} \\
&- \frac{3\,{\rho'_A}^2 \left(2\,{\rho_A}^4 - 4\,{\rho_A}^3 + 8\,\rho_A - 8\right)}{{\rho'_A}^2 \left(2\,{\rho_A}^4 - 2\,{\rho_A}^3 + 4\,{\rho_A}^2\right) - 8\,\rho'_A\,{\rho_A}^3 + 4\,{\rho_A}^3} \\
&+ \frac{2\,\rho'_A \left(-{\rho_A}^4 - {\rho_A}^3 + 2\,\rho_A + 4\right) + 2\,{\rho_A}^3 - 4\,\rho_A}{{\rho'_A}^2 \left(2\,{\rho_A}^4 - 2\,{\rho_A}^3 + 4\,{\rho_A}^2\right) - 8\,\rho'_A\,{\rho_A}^3 + 4\,{\rho_A}^3}.
\end{aligned}$$

*One can show that this expression is bigger than 1, proving that $h_1(\rho'_A, q'_A)$ is strictly increasing with respect to ρ'_A, which implies that **(H-2)** holds.*

For $(\rho, q) \in \Omega_f \cup \Omega_c$, define the following sets

$$\mathcal{T}_l^2(\rho, q) = \begin{cases} \left\{(\tilde\rho, \tilde q) \in \Omega_c : \tilde q = \frac{q}{\rho}\tilde\rho\right\}, & \text{if } (\rho, q) \in \Omega_c, \\ (\rho, q) \cup \left\{(\tilde\rho, \tilde q) \in \Omega_c : \begin{array}{c} \tilde q = \frac{q^-}{R}\tilde\rho, \\ \tilde\rho v_c(\tilde\rho, \tilde q) < V\rho \end{array}\right\}, & \text{if } (\rho, q) \in \Omega_f \setminus \Omega_c, \end{cases} \tag{12}$$

and

$$\mathcal{T}_r^2(\rho, q) = \begin{cases} \Omega_f, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{array}{l} \{(\tilde\rho, \tilde q) \in \Omega_c : v_c^{eq}(\tilde\rho)(1 + \tilde q) = v_c^{eq}(\rho)(1 + q)\} \\ \cup \left\{(\tilde\rho, \tilde q) \in \Omega_f : \tilde\rho V < \varphi(\psi_2^-(\rho, q))\right\}, \end{array} & \text{if } (\rho, q) \in \Omega_c, \end{cases} \tag{13}$$

Remark 2 *Note that, if $(\rho, q) \in \Omega_c$, then the sets*

$$\left\{(\tilde\rho, \tilde q) \in \Omega_c : \tilde q = \frac{q}{\rho}\tilde\rho\right\} \qquad \text{and} \qquad \{(\tilde\rho, \tilde q) \in \Omega_c : v_c^{eq}(\tilde\rho)(1 + \tilde q) = v_c^{eq}(\rho)(1 + q)\},$$

appearing in (12) and in (13), are composed by all the points in Ω_c *which can be connected to* (ρ, q) *by a Lax curve respectively of the first family and by the second one. Moreover the set*

$$\left\{(\tilde\rho, \tilde q) \in \Omega_c : \tilde q = \frac{q^-}{R}\tilde\rho,\ \tilde\rho v_c(\tilde\rho, \tilde q) < V\rho\right\}$$

in (12) contains all the points in Ω_c *of the first Lax curve passing through* (R, q^-) *with sufficiently small flux. Finally, the set* $\left\{(\tilde\rho, \tilde q) \in \Omega_f : \tilde\rho V < \varphi(\psi_2^-(\rho, q))\right\}$ *consists in all the points of* Ω_f *with flux less than* $\varphi(\psi_2^-(\rho, q))$.

The following lemma holds.

Lemma 3.1 *We have that*

$$\varphi\left(\mathcal{T}_l^2(\rho, q)\right) = \begin{cases} [0, \varphi(\psi_1(\rho, q))], & \text{if } (\rho, q) \in \Omega_c, \\ [0, \rho V], & \text{if } (\rho, q) \in \Omega_f \setminus \Omega_c, \end{cases} \tag{14}$$

and

$$\varphi\left(\mathcal{T}_r^2(\rho, q)\right) = \begin{cases} [0, \varphi(\psi_2^+(\rho, q))], & \text{if } (\rho, q) \in \Omega_c, \\ [0, \sigma_+ V], & \text{if } (\rho, q) \in \Omega_f. \end{cases} \tag{15}$$

Proof. First note that, in the $(\rho, \rho v_c(\rho))$ plane, the Lax curves in Ω_c can be described by the graphs of strictly monotone functions.

In fact, the Lax curves of the first family in the $(\rho, \rho v_c(\rho))$ plane are given by

$$\left(\rho, \frac{V\sigma}{R-\sigma}(R-\rho)(1+c\rho)\right),$$

where $c \in \left[\frac{q^-}{R}, \frac{q^+}{R}\right]$, $0 \le \rho \le R$ and $v_c(\rho, c\rho) \le V$; hence they can be described by the graph of the function $h(\rho) = \frac{V\sigma}{R-\sigma}(R-\rho)(1+c\rho)$. We claim that $h'(\rho) = \frac{V\sigma}{R-\sigma}(cR - 1 - 2c\rho) < 0$ for every $0 \le \rho \le R$ such that $v_c(\rho, c\rho) \le V$. Since h' is affine in ρ and $h'(R) = \frac{V\sigma}{R-\sigma}(-1 - cR) < 0$, it is sufficient to show that $h'(\bar\rho) < 0$, where $\bar\rho$ satisfies

$$\begin{cases} (\bar\rho, c\bar\rho) \in \Omega_c, \\ v_c(\bar\rho, c\bar\rho) = V. \end{cases}$$

With simple computations we deduce that

$$\bar\rho = \frac{-R(1-\sigma c) + \sqrt{R^2(1+\sigma^2c^2 - 2\sigma c) + 4\sigma^2 Rc}}{2\sigma c}$$

and $h'(\bar\rho) < 0$ if and only if

$$R^2\sigma c^2 + 2R(2\sigma - R)c + 2R - \sigma > 0.$$

Since $\frac{R}{5} < \sigma < R$, the last inequality is satisfied for every $c \in \mathbb{R}$ and so we conclude for the Lax curves of the first family.

The Lax curves of the second family in the $(\rho, \rho v_c(\rho))$ plane are lines through the origin and so the conclusion easily follows.

Let us prove (14). If $(\rho, q) \in \Omega_c$, then the set $\mathcal{T}_l^2(\rho, q)$ is composed by all the points in Ω_c belonging to the Lax curve of the first family passing through (ρ, q). By definition of the function φ and by the previous observations on the Lax curves, we deduce that $\varphi\left(\mathcal{T}_l^2(\rho, q)\right) = [0, \varphi(\psi_1(\rho, q))]$.
If $(\rho, q) \in \Omega_f \setminus \Omega_c$, then the set $\mathcal{T}_l^2(\rho, q)$ is composed by (ρ, q) and by all the points in Ω_c belonging to the Lax curve of the first family passing through (R, q^-) with flux lower than $\varphi(\rho, q)$. As in the previous case, we deduce that $\varphi\left(\mathcal{T}_l^2(\rho, q)\right) = [0, \varphi(\rho, q)] = [0, \rho V]$.

Let us prove (15). If $(\rho, q) \in \Omega_f$, then the set $\mathcal{T}_r^2(\rho, q)$ is composed by all the points in Ω_f. Clearly $\varphi\left(\mathcal{T}_r^2(\rho, q)\right) = \varphi(\Omega_f) = [0, \sigma_+ V]$.
If $(\rho, q) \in \Omega_c$, then the set $\mathcal{T}_r^2(\rho, q)$ is composed by all the points in Ω_c belonging to the Lax curve of the second family passing through (ρ, q) and by all the point in Ω_f with flux less than $\varphi(\psi_2^-(\rho, q))$. By definition of the function φ and by the previous observations on the Lax curves, we deduce that $\varphi\left(\mathcal{T}_r^2(\rho, q)\right) = [0, \varphi(\psi_2^+(\rho, q))]$.
□

Lemma 3.2 *The first eigenvalue λ_1 is strictly negative in Ω_c. The second eigenvalue λ_2 is positive in Ω_c.*

PROOF. By (9), the first eigenvalue can be written in the form

$$\lambda_1(\rho,q) = \frac{V\sigma}{R-\sigma}\left[(1+2q)\left(\frac{R}{\rho}-1\right)-(1+q)\frac{R}{\rho}\right]$$

and so

$$\frac{\partial}{\partial\rho}\lambda_1(\rho,q) = -\frac{V\sigma Rq}{\rho^2(R-\sigma)}, \quad \frac{\partial}{\partial q}\lambda_1(\rho,q) = \frac{V\sigma}{R-\sigma}\left(\frac{R}{\rho}-2\right)$$

The only critical point is $\left(\frac{R}{2},0\right)$. Note that $\left(\frac{R}{2},0\right)\in\Omega_c$ if and only if $\sigma\le\frac{R}{2}$. However $\lambda_1\left(\frac{R}{2},0\right) = -\frac{V\sigma}{R-\sigma}<0$. Therefore it is sufficient to prove that λ_1 is strictly negative on the boundary of Ω_c.
If $\rho = R$, then $\lambda_1 = -\frac{V\sigma}{R-\sigma}(1+q)$, which is strictly negative.
If $\frac{q}{\rho} = \frac{q^-}{R}$, then

$$\lambda_1 = \frac{V\sigma}{R-\sigma}\left(q^- - 1 - \frac{2q^-}{R}\rho\right) \le \frac{V\sigma}{R-\sigma}\left(q^- - 1\right),$$

which is strictly negative.
If $v_c(\rho,q) = V$, then

$$\lambda_1 = \frac{V\sigma}{R-\sigma}\,\frac{R^2\rho - R^2\sigma - 2R\rho^2 + \sigma\rho^2 + R\sigma\rho}{\sigma\rho(R-\rho)},$$

which is strictly negative if and only if

$$h(\rho) := (2R-\sigma)\,\rho^2 - R\,(R+\sigma)\,\rho + \sigma R^2 > 0$$

for every $\rho\in[\sigma_-,\sigma_+]$. The minimum point for h is equal to $\rho = \frac{R(R+\sigma)}{2(2R-\sigma)}$ and

$$h\left(\frac{R(R+\sigma)}{2(2R-\sigma)}\right) = -\frac{R^2(R+\sigma)^2}{4(2R-\sigma)} + \sigma R^2$$

which is strictly positive, since $\frac{R}{5}<\sigma<R$.
If $\frac{q}{\rho} = \frac{q^+}{R}$, then

$$\lambda_1 = \frac{V\sigma}{R-\sigma}\left(q_+ - 1 - \frac{2q^+}{R}\rho\right) \le \frac{V\sigma}{R-\sigma}\left(q^+ - 1 - \frac{2q^+}{R}\sigma_+\right),$$

which is strictly negative, by the previous computation. Therefore the first eigenvalue λ_1 is strictly negative in Ω_c.

The fact that second eigenvalue λ_2 is positive in Ω_c is trivial. The proof is so concluded. $\square$

Proposition 3.1 *Since $-1<q^-<0<q^+<1$, the Lax curves of the first family in the $(\rho,\rho v)$ plane, parametrized by ρ, are uniformly bi-Lipschitz in Ω_c. For every $0<\bar v<V$, the Lax curves of the second family in the $(\rho,\rho v)$ plane, parametrized by ρ, are uniformly bi-Lipschitz in the set $\{(\rho,q)\in\Omega_c : v_c(\rho,q)\ge\bar v\}$.*

PROOF. The Lax curves of the first family in Ω_c with respect to the $(\rho,\rho v)$-coordinates can be described as the graph of the map

$$\rho\mapsto\frac{V\sigma}{R-\sigma}\,(R-\rho)\left(1+\frac{q_0}{\rho_0}\rho\right), \tag{16}$$

where (ρ_0,q_0) denotes the starting point. The derivative of (16) is given by

$$\rho\mapsto\frac{V\sigma}{R-\sigma}\left(-1-2\frac{q_0}{\rho_0}\rho+\frac{q_0}{\rho_0}R\right). \tag{17}$$

If $q_0>0$, then (17) is strictly decreasing. For $\rho=0$, the derivative (17) becomes $\frac{V\sigma}{R-\sigma}\left(-1+\frac{q_0}{\rho_0}R\right)\le\frac{V\sigma}{R-\sigma}\left(-1+q^+\right)<0$ by (5) and by $q^+<1$. For $\rho=R$, the derivative (17) is uniformly bounded and negative.
If $q_0<0$, then (17) is strictly increasing. For $\rho=0$, the derivative (17) is uniformly bounded and less than the value of (17) at $\rho=R$. For $\rho=R$ (17) becomes $\frac{V\sigma}{R-\sigma}\left(-1-\frac{q_0}{\rho_0}R\right)\le\frac{V\sigma}{R-\sigma}\left(-1-q^-\right)<0$ by (5). Therefore the conclusion for the Lax curves of the first family follows.

The Lax curves of the second family in Ω_c with respect to the $(\rho,\rho v)$-coordinates are lines passing through the origin with angular coefficient between $\bar v$ and V and so the conclusion easily follows. $\square$

We have the following characterization of the sets $\mathcal{T}_l^2(\rho,q)$ and $\mathcal{T}_r^2(\rho,q)$.

Proposition 3.2 *For $(\rho_l, q_l) \in \Omega_f \cup \Omega_c$ and $(\rho_r, q_r) \in \Omega_f \cup \Omega_c$, consider the Riemann problem*

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho, q)) = 0, & \text{if } (\rho, q) \in \Omega_f,\ t > 0, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho, q)) = 0, \end{cases} & \text{if } (\rho, q) \in \Omega_c,\ t > 0, \\ (\rho, q)(0, x) = \begin{cases} (\rho_l, q_l), & \text{if } x < 0, \\ (\rho_r, q_r), & \text{if } x > 0. \end{cases} \end{cases} \tag{18}$$

1. *The Riemann problem (18) is solved with waves with non positive speed if and only if $(\rho_r, q_r) \in \mathcal{T}_l^2(\rho_l, q_l)$.*
2. *The Riemann problem (18) is solved with waves with non negative speed if and only if $(\rho_l, q_l) \in \mathcal{T}_r^2(\rho_r, q_r)$.*

Proof. Consider first point 1. We have two different situations.

1. $(\rho_l, q_l) \in \Omega_f \setminus \Omega_c$. If $(\rho_r, q_r) \in \Omega_f$ and $(\rho_r, q_r) \neq (\rho_l, q_l)$, then a contact discontinuity wave with velocity $V > 0$ appears. If $(\rho_r, q_r) \in \Omega_c$ with $\frac{q_r}{\rho_r} > \frac{q^-}{R}$, then a contact discontinuity wave of the second family with speed $v_c(\rho_r, q_r) > 0$ appears. If $(\rho_r, q_r) \in \Omega_c$ satisfies $\frac{q_r}{\rho_r} = \frac{q^-}{R}$ and $\varphi(\rho_r, q_r) \geq \varphi(\rho_l, q_l)$, then a phase transition wave with velocity greater than or equal to 0 appears. Finally, if $(\rho_r, q_r) \in \Omega_c$ satisfies $\frac{q_r}{\rho_r} = \frac{q^-}{R}$ and $\varphi(\rho_r, q_r) < \varphi(\rho_l, q_l)$, then the Riemann problem (18) is solved with a phase transition wave with strictly negative speed, possibly coupled with a wave of the first family.
   Therefore the Riemann problem (18) is solved with waves with non positive speed if and only if either $(\rho_r, q_r) = (\rho_l, q_l)$ or (ρ_r, q_r) belongs to the set
   $$\left\{ (\tilde{\rho}, \tilde{q}) \in \Omega_c : \tilde{q} = \frac{q^-}{R} \tilde{\rho},\ \ \varphi(\tilde{\rho}, \tilde{q}) < \varphi(\rho_l, q_l) \right\}.$$
2. $(\rho_l, q_l) \in \Omega_c$. In this case the Riemann problem (18) is solved with waves with non positive speed if and only if (ρ_l, q_l) and (ρ_r, q_r) are connected by a wave of the first family, i.e.
   $$\begin{cases} (\rho_r, q_r) \in \Omega_c, \\ \frac{q_l}{\rho_l} = \frac{q_r}{\rho_r}. \end{cases}$$

This concludes the proof of point 1.
Consider now point 2. In this case no wave of the first family should appear. We have two situations.

1. $(\rho_r, q_r) \in \Omega_f$. If $(\rho_l, q_l) \in \Omega_c \setminus \Omega_f$, then the Riemann problem (18) is solved by using a wave of the first family with strictly negative speed. If $(\rho_l, q_l) \in \Omega_f$, then the Riemann problem (18) is solved with a contact discontinuity traveling with speed $V > 0$.
   Therefore, in this case the Riemann problem (18) is solved with waves with non negative speed if and only if $(\rho_l, q_l) \in \Omega_f$.
2. $(\rho_r, q_r) \in \Omega_c$. If $(\rho_l, q_l) \in \Omega_c$ satisfies $v_c(\rho_l, q_l) = v_c(\rho_r, q_r)$, then the solution to the Riemann problem (18) consists of a wave of the second family with speed $v_c(\rho_l, q_l) > 0$.
   If $(\rho_l, q_l) \in \Omega_c$ satisfies $v_c(\rho_l, q_l) \neq v_c(\rho_r, q_r)$, then in the solution to the Riemann problem (18) a wave of the first family with negative speed appears.
   If $(\rho_l, q_l) \in \Omega_f \setminus \Omega_c$ satisfies $\varphi(\rho_l, q_l) \geq \varphi(\psi_2^-(\rho_r, q_r))$, then the solution to (18) contains a phase transition wave with speed lower than or equal to 0.
   Finally, If $(\rho_l, q_l) \in \Omega_f \setminus \Omega_c$ satisfies $\varphi(\rho_l, q_l) < \varphi(\psi_2^-(\rho_r, q_r))$, then the solution to (18) consists of a phase transition wave with strictly positive speed and of a wave of the second family.

   Hence the Riemann problem (18) is solved by waves with non negative speed if and only if (ρ_l, q_l) belongs to
   $$\{(\tilde{\rho}, \tilde{q}) \in \Omega_c : v_c(\tilde{\rho}, \tilde{q}) = v_c(\rho_r, q_r)\}$$
   or to
   $$\left\{(\tilde{\rho}, \tilde{q}) \in \Omega_f : \varphi(\tilde{\rho}, \tilde{q}) < \varphi(\psi_2^-(\rho_r, q_r))\right\}.$$

This concludes the proof of the point 2 and so the proof of the proposition. □

4 The LWR-PT model

This section deals with the coupling between the LWR model if $x < 0$ and the PT one if $x > 0$. More precisely we consider the following system

$$\begin{cases} \partial_t \rho + \partial_x(\rho \tilde{v}(\rho)) = 0, & \text{if } x < 0,\ t > 0, \\ \begin{cases} \partial_t \rho + \partial_x(\rho V) = 0, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho, q)) = 0. \end{cases} & \text{if } (\rho, q) \in \Omega_c, \end{cases} & \text{if } x > 0,\ t > 0. \end{cases} \tag{19}$$

Definition 4.1 *The functions*

$$\hat{\rho}_l \in C([0, +\infty[; L^1_{loc}(] - \infty, 0]; [0, R])) \qquad \textit{and} \qquad (\hat{\rho}_r, \hat{q}_r) \in C([0, +\infty[; L^1_{loc}([0, +\infty[; \Omega_f \cup \Omega_c))$$

are a weak solution to (19) if

1. *the function $\hat{\rho}_l$ is a weak solution to*
$$\partial_t \rho + \partial_x(\rho \tilde{v}(\rho)) = 0$$
for $(t, x) \in (0, +\infty) \times (-\infty, 0)$;
2. *the function $(\hat{\rho}_r, \hat{q}_r)$ is a weak solution to*
$$\begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho, q)) = 0, & \textit{if } (\rho, q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho, q)) = 0. \end{cases} & \textit{if } (\rho, q) \in \Omega_c, \end{cases}$$
for $(t, x) \in (0, +\infty) \times (0, +\infty)$;
3. *for a.e. $t > 0$, the functions $x \mapsto \hat{\rho}_l(t, x)$ and $x \mapsto (\hat{\rho}_r(t, x), \hat{q}_r(t, x))$ both have versions with bounded total variation;*
4. *for a.e. $t > 0$, it holds*
$$f(\hat{\rho}_l(t, 0-)) = \varphi\left(\hat{\rho}_r(t, 0+), \hat{q}_r(t, 0+)\right),$$
where $\hat{\rho}_l$ and $(\hat{\rho}_r, \hat{q}_r)$ stand for the versions with bounded total variation.

For functions

$$\begin{cases} \hat{\rho}_l \in C([0, +\infty[; L^1_{loc}(] - \infty, 0])) \\ (\hat{\rho}_r, \hat{q}_r) \in C([0, +\infty[; L^1_{loc}([0, +\infty[)) \end{cases}$$

such that for a.e. $t > 0$ the maps $x \mapsto \hat{\rho}_l(t, x)$ and $x \mapsto (\hat{\rho}_r(t, x), \hat{q}_r(t, x))$ both have versions with bounded total variation, we introduce the functional

$$\text{Tot.Var.}^f_{1,2}(t) = \text{Tot.Var.} f(\hat{\rho}_l(t, \cdot)) + \text{Tot.Var.} \varphi(\hat{\rho}_r(t, \cdot), \hat{q}_r(t, \cdot)) \tag{20}$$

which is clearly defined for a.e. $t > 0$.

4.1 The Riemann Problem for LWR-PT coupled model

In this section we describe the Riemann problem for the LWR-PT model. More precisely we consider system (19) with the piecewise constant initial conditions

$$\begin{cases} \rho(0, x) = \rho_l, & \text{if } x < 0, \\ (\rho(0, x), q(0, x)) = (\rho_r, q_r), & \text{if } x > 0, \end{cases} \tag{21}$$

where $\rho_l \in [0, R]$ and $(\rho_r, q_r) \in \Omega_f \cup \Omega_c$.

Giving a solution to (19)-(21) is equivalent to prescribe a left and right trace at $x = 0$, denoted respectively by ρ_l^b and (ρ_r^b, q_r^b). So we need to find $\rho_l^b \in \mathcal{T}_l^1(\rho_l)$ and $(\rho_r^b, q_r^b) \in \mathcal{T}_r^2(\rho_r, q_r)$ satisfying suitable compatibility conditions.

First of all, we require the conservation of the number of vehicles passing through the boundary $x = 0$. The number of cars exiting from the region $x < 0$ per unit time is given by

$$\rho_l^b \tilde{v}(\rho_l^b) = \rho_l^b V_{max} \left(1 - \frac{\rho_l^b}{R}\right). \tag{22}$$

The number of vehicles entering in the region $x > 0$ per unit time is given by

$$\varphi(\rho_r^b, q_r^b) = \begin{cases} V\rho_r^b, & \text{if } (\rho_r^b, q_r^b) \in \Omega_f, \\ \frac{V\sigma}{R-\sigma}\rho_r^b \left(\frac{R}{\rho_r^b} - 1\right)\left(1 + q_r^b\right) & \text{if } (\rho_r^b, q_r^b) \in \Omega_c. \end{cases} \tag{23}$$

Thus condition 4. of Definition 4.1 becomes

$$\rho_l^b V_{max} \left(1 - \frac{\rho_l^b}{R}\right) = \begin{cases} V\rho_r^b, & \text{if } (\rho_r^b, q_r^b) \in \Omega_f \\ \frac{V\sigma}{R-\sigma}\rho_r^b \left(\frac{R}{\rho_r^b} - 1\right)\left(1 + q_r^b\right), & \text{if } (\rho_r^b, q_r^b) \in \Omega_c. \end{cases} \tag{24}$$

Let us now define the concept of Riemann solver for (19)-(21).

Definition 4.2 *A Riemann solver for the Riemann problem (19)-(21) is a function*

$$\begin{aligned} \mathcal{RS}_{1,2} : [0,R] \times (\Omega_f \cup \Omega_c) &\longrightarrow [0,R] \times (\Omega_f \cup \Omega_c) \\ (\rho_l, (\rho_r, q_r)) &\longmapsto (\rho_l^b, (\rho_r^b, q_r^b)) \end{aligned}$$

such that:

1. $\rho_l^b \in \mathcal{T}_l^1(\rho_l)$ *and* $(\rho_r^b, q_r^b) \in \mathcal{T}_r^2(\rho_r, q_r)$;
2. $\rho_l^b V_{max}\left(1 - \frac{\rho_l^b}{R}\right) = \varphi(\rho_r^b, q_r^b)$;
3. $\mathcal{RS}_{1,2}(\mathcal{RS}_{1,2}(\rho_l, (\rho_r, q_r))) = \mathcal{RS}_{1,2}(\rho_l, (\rho_r, q_r))$.

Remark 3 *The image of a Riemann solver $\mathcal{RS}_{1,2}$ gives the traces at $x = 0-$ and at $x = 0+$ of a solution to the Riemann problem (19)-(21). Condition 1. of Definition 4.2 implies that the waves produced to connect ρ_l to ρ_l^b and (ρ_r^b, q_r^b) to (ρ_r, q_r) have respectively negative and positive speed. Condition 2. of Definition 4.2 implies that the number of cars are conserved at the interface, while condition 3. is the consistency condition.*

Consider the maximum flow, which can pass through the interface $x = 0$,

$$\Gamma_{\rho_l,(\rho_r,q_r)} = \min\left\{ \sup_{\rho \in \mathcal{T}_l^1(\rho_l)} \rho V_{max}\left(1 - \frac{\rho}{R}\right), \sup_{(\rho,q)\in\mathcal{T}_r^2(\rho_r,q_r)} \varphi(\rho, q) \right\}. \tag{25}$$

The following lemma holds.

Lemma 4.1 *Fix $\gamma \in [0, \Gamma_{\rho_l,(\rho_r,q_r)}]$, where $\Gamma_{\rho_l,(\rho_r,q_r)}$ is defined in (25). There exist a unique $\rho_l^b \in \mathcal{T}_l^1(\rho_l)$ and a unique $(\rho_r^b, q_r^b) \in \mathcal{T}_r^2(\rho_r, q_r)$ such that*

$$\rho_l^b V_{max}\left(1 - \frac{\rho_l^b}{R}\right) = \gamma$$

and

$$\varphi(\rho_r^b, q_r^b) = \gamma.$$

Proof. The lemma is based on the fact that the function φ is injective on $\mathcal{T}_r^2(\rho_r, q_r)$ and that the function $\rho \mapsto \rho V_{max}\left(1 - \frac{\rho}{R}\right)$ is injective on $\mathcal{T}_l^1(\rho_l)$, as seen in the proof of Lemma 3.1. □

For the LWR-PT model we introduce the following definitions.

Definition 4.3 *Given a Riemann solver $\mathcal{RS}_{1,2}$, we say that $(\rho_l, (\rho_r, q_r))$ is an equilibrium for $\mathcal{RS}_{1,2}$ if*

$$\mathcal{RS}_{1,2}(\rho_l, (\rho_r, q_r)) = (\rho_l, (\rho_r, q_r)).$$

Definition 4.4 *Consider an equilibrium $(\rho_l, (\rho_r, q_r))$ for $\mathcal{RS}_{1,2}$. We say that ρ_l provides a constraint for $(\rho_l, (\rho_r, q_r))$ if*

$$f(\rho_l) = \sup_{\rho\in\mathcal{T}_l^1(\rho_l)} f(\rho).$$

We say that (ρ_r, q_r) provides a constraint for $(\rho_l, (\rho_r, q_r))$ if

$$\varphi(\rho_r, q_r) = \sup_{(\rho,q)\in\mathcal{T}_r^2(\rho_r,q_r)} \varphi(\rho, q).$$

We now characterize two different Riemann solvers for (19)-(21).

4.1.1 The Riemann solver which maximizes the flux

Here we construct the Riemann solver $\mathcal{RS}^1_{1,2}$ for (19)-(21), which maximizes the flux passing through the interface. The construction is done in the following way.

1. Given $\rho_l \in [0, R]$ and $(\rho_r, q_r) \in \Omega_f \cup \Omega_c$, define the maximum flow $\Gamma_{\rho_l,(\rho_r,q_r)}$ as in equation (25).

2. By Lemma 4.1, there exist a unique $\rho_l^b \in \mathcal{T}_l^1(\rho_l)$ and a unique $(\rho_r^b, q_r^b) \in \mathcal{T}_r^2(\rho_r, q_r)$ such that

$$\rho_l^b V_{max}\left(1 - \frac{\rho_l^b}{R}\right) = \Gamma_{\rho_l,(\rho_r,q_r)} \quad \text{and} \quad \varphi(\rho_r^b, q_r^b) = \Gamma_{\rho_l,(\rho_r,q_r)}.$$

3. Define $\mathcal{RS}^1_{1,2}(\rho_l, (\rho_r, q_r)) = \left(\rho_l^b, (\rho_r^b, q_r^b)\right)$.

The following proposition holds.

Proposition 4.1 *The function $\mathcal{RS}^1_{1,2}$ is a Riemann solver for the Riemann problem (19)-(21).*

Proof. By construction, the function $\mathcal{RS}^1_{1,2}$ satisfies the first two properties of Definition 4.2. Thus it remains to prove that $\mathcal{RS}^1_{1,2}\left(\mathcal{RS}^1_{1,2}(\rho_l, (\rho_r, q_r))\right) = \mathcal{RS}^1_{1,2}(\rho_l, (\rho_r, q_r))$ for every $\rho_l \in [0, R]$ and $(\rho_r, q_r) \in \Omega_f \cup \Omega_c$. Define $\left(\rho_l^b, (\rho_r^b, q_r^b)\right) = \mathcal{RS}^1_{1,2}(\rho_l, (\rho_r, q_r))$. We claim that

$$\Gamma_{\rho_l,(\rho_r,q_r)} = \Gamma_{\rho_l^b,(\rho_r^b,q_r^b)}. \tag{26}$$

We have two possibilities.

1. $\Gamma_{\rho_l,(\rho_r,q_r)} = \sup_{\rho \in \mathcal{T}_l^1(\rho_l)} \rho V_{max}\left(1 - \frac{\rho}{R}\right)$. In this case $\mathcal{T}_l^1(\rho_l) = \mathcal{T}_l^1\left(\rho_l^b\right)$ and so

$$\sup_{\rho \in \mathcal{T}_l^1(\rho_l)} \rho V_{max}\left(1 - \frac{\rho}{R}\right) = \sup_{\rho \in \mathcal{T}_l^1\left(\rho_l^b\right)} \rho V_{max}\left(1 - \frac{\rho}{R}\right).$$

Moreover, if $(\rho_r, q_r), (\rho_r^b, q_r^b) \in \Omega_f$ or $(\rho_r, q_r), (\rho_r^b, q_r^b) \in \Omega_c$, then

$$\sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r,q_r)} \varphi(\rho, q) = \sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r^b,q_r^b)} \varphi(\rho, q)$$

and so (26) holds. If $(\rho_r, q_r) \in \Omega_c$ and $(\rho_r^b, q_r^b) \in \Omega_f$, then

$$\sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r,q_r)} \varphi(\rho, q) < \sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r^b,q_r^b)} \varphi(\rho, q)$$

and so (26) holds. Since the case $(\rho_r, q_r) \in \Omega_f$ and $(\rho_r^b, q_r^b) \in \Omega_c \setminus \Omega_f$ does not happen, then we conclude that

$$\sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r,q_r)} \varphi(\rho, q) \leq \sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r^b,q_r^b)} \varphi(\rho, q)$$

and so (26) holds.

2. $\Gamma_{\rho_l,(\rho_r,q_r)} = \sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r,q_r)} \varphi(\rho, q)$. In this case $\mathcal{T}_r^2(\rho_r, q_r) = \mathcal{T}_r^2\left(\rho_r^b, q_r^b\right)$ and so

$$\sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r,q_r)} \varphi(\rho, q) = \sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r^b,q_r^b)} \varphi(\rho, q).$$

If $\rho_l = \rho_l^b$, then $\mathcal{T}_l^1(\rho_l) = \mathcal{T}_l^1\left(\rho_l^b\right)$ and so

$$\sup_{\rho \in \mathcal{T}_l^1(\rho_l)} \rho V_{max}\left(1 - \frac{\rho}{R}\right) = \sup_{\rho \in \mathcal{T}_l^1\left(\rho_l^b\right)} \rho V_{max}\left(1 - \frac{\rho}{R}\right).$$

If $\frac{R}{2} \leq \rho_l < \rho_l^b$, then, by (3), $\mathcal{T}_l^1(\rho_l) = \mathcal{T}_l^1\left(\rho_l^b\right) = \left[\frac{R}{2}, R\right]$ and so

$$\sup_{\rho \in \mathcal{T}_l^1(\rho_l)} \rho V_{max}\left(1 - \frac{\rho}{R}\right) = \sup_{\rho \in \mathcal{T}_l^1\left(\rho_l^b\right)} \rho V_{max}\left(1 - \frac{\rho}{R}\right).$$

If $\rho_l < \rho_l^b$ and $\rho_l < \frac{R}{2}$, then, by (3), $\mathcal{T}_l^1(\rho_l) = \{\rho_l\} \cup]R - \rho_l, R]$. Since $\rho_l^b \in \mathcal{T}_l^1(\rho_l)$ and $\rho_l^b > \rho_l$, then $\rho_l^b > R - \rho_l > \frac{R}{2}$ and so $\mathcal{T}_l^1(\rho_l^b) = \left[\frac{R}{2}, R\right]$; hence

$$\sup_{\rho \in \mathcal{T}_l^1(\rho_l)} \rho V_{max}\left(1 - \frac{\rho}{R}\right) < \sup_{\rho \in \mathcal{T}_l^1(\rho_l^b)} \rho V_{max}\left(1 - \frac{\rho}{R}\right)$$

and (26) holds. Finally, if $\rho_l > \rho_l^b$, then $\rho_l \geq \frac{R}{2}$. Indeed $\rho_l < \frac{R}{2}$ is not possible by (3). Therefore $\mathcal{T}_l^1(\rho_l) = \left[\frac{R}{2}, R\right]$, $\rho_l^b \geq \frac{R}{2}$ and $\mathcal{T}_l^1(\rho_l) = \mathcal{T}_l^1(\rho_l^b)$; hence

$$\sup_{\rho \in \mathcal{T}_l^1(\rho_l)} \rho V_{max}\left(1 - \frac{\rho}{R}\right) = \sup_{\rho \in \mathcal{T}_l^1(\rho_l^b)} \rho V_{max}\left(1 - \frac{\rho}{R}\right).$$

Thus we deduce that (26) holds.

Note that $\rho_l^b \in \mathcal{T}_l^1(\rho_l^b)$, $(\rho_r^b, q_r^b) = \mathcal{T}_r^2(\rho_r^b, q_r^b)$ and

$$\rho_l^b V_{max}\left(1 - \frac{\rho_l^b}{R}\right) = \Gamma_{\rho_l,(\rho_r,q_r)} = \varphi(\rho_r^b, q_r^b).$$

Therefore Lemma 4.1 permits to conclude. □

4.1.2 The Riemann solver with a flux constraint

Fix a positive constant $\bar{k} > 0$. Here we construct a Riemann solver $\mathcal{RS}_{1,2}^2$ for (19)-(21), which imposes a constraint on the flux passing through the interface. The construction is done in the following way.

1. Given $\rho_l \in [0, R]$ and $(\rho_r, q_r) \in \Omega_f \cup \Omega_c$, define the maximum flow $\Gamma_{\rho_l,(\rho_r,q_r)}$ as in equation (25).
2. By Lemma 4.1, there exist a unique $\rho_l^b \in \mathcal{T}_l^1(\rho_l)$ and a unique $(\rho_r^b, q_r^b) \in \mathcal{T}_r^2(\rho_r, q_r)$ such that

$$\rho_l^b V_{max}\left(1 - \frac{\rho_l^b}{R}\right) = \min\left\{\bar{k}, \Gamma_{\rho_l,(\rho_r,q_r)}\right\}$$

   and

$$\varphi(\rho_r^b, q_r^b) = \min\left\{\bar{k}, \Gamma_{\rho_l,(\rho_r,q_r)}\right\}.$$

3. Define $\mathcal{RS}_{1,2}^2(\rho_l, (\rho_r, q_r)) = \left(\rho_l^b, (\rho_r^b, q_r^b)\right)$.

The following proposition holds.

Proposition 4.2 *Given $\bar{k} > 0$, the function $\mathcal{RS}_{1,2}^2$ is a Riemann solver for the Riemann problem (19)-(21).*

Proof. By construction, the function $\mathcal{RS}_{1,2}^2$ satisfies the first two properties of Definition 4.2. Thus it remains to prove that $\mathcal{RS}_{1,2}^2\left(\mathcal{RS}_{1,2}^2(\rho_l, (\rho_r, q_r))\right) = \mathcal{RS}_{1,2}^2(\rho_l, (\rho_r, q_r))$ for every $\rho_l \in [0, R]$ and $(\rho_r, q_r) \in \Omega_f \cup \Omega_c$. Define the state $\left(\rho_l^b, (\rho_r^b, q_r^b)\right)$ equal to $\mathcal{RS}_{1,2}^2(\rho_l, (\rho_r, q_r))$. We have two different possibilities.

1. $\Gamma_{\rho_l,(\rho_r,q_r)} \leq \bar{k}$. With the same arguments as in the proof of Proposition 4.1, we deduce that $\Gamma_{\rho_l,(\rho_r,q_r)} = \Gamma_{\rho_l^b,(\rho_r^b,q_r^b)} \leq \bar{k}$ and so $\left(\rho_l^b, (\rho_r^b, q_r^b)\right) = \mathcal{RS}_{1,2}^2\left(\rho_l^b, (\rho_r^b, q_r^b)\right)$.
2. $\Gamma_{\rho_l,(\rho_r,q_r)} > \bar{k}$. In this case we easily deduce that

$$\sup_{\rho \in \mathcal{T}_l^1(\rho_l)} \rho V_{max}\left(1 - \frac{\rho}{R}\right) \leq \sup_{\rho \in \mathcal{T}_l^1(\rho_l^b)} \rho V_{max}\left(1 - \frac{\rho}{R}\right)$$

   and

$$\sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r,q_r)} \varphi(\rho, q) \leq \sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r^b,q_r^b)} \varphi(\rho, q).$$

   Therefore $\Gamma_{\rho_l^b,(\rho_r^b,q_r^b)} > \bar{k}$ and so $\left(\rho_l^b, (\rho_r^b, q_r^b)\right) = \mathcal{RS}_{1,2}^2\left(\rho_l^b, (\rho_r^b, q_r^b)\right)$.

The proof is finished. □

4.2 The Cauchy problem

This subsection deals with the Cauchy problem for the LWR-PT coupled model. Fix $\rho_l \in BV(]-\infty,0];[0,R])$, $(\rho_r,q_r) \in BV([0,+\infty[;\Omega_f \cup \Omega_c)$ and a Riemann solver $\mathcal{RS}_{1,2}$, which is either $\mathcal{RS}^1_{1,2}$ or $\mathcal{RS}^2_{1,2}$. Consider the Cauchy problem for (19) with the initial condition

$$\begin{cases} \rho(0,x) = \rho_l(x), & \text{if } x < 0, \\ (\rho(0,x), q(0,x)) = (\rho_r(x), q_r(x)), & \text{if } x > 0. \end{cases} \tag{27}$$

The main result for the LWR-PT model is the following theorem. The proof is contained in the next subsections.

Theorem 4.1 *Fix $\rho_l \in BV(]-\infty,0];[0,R])$ and $(\rho_r,q_r) \in BV([0,+\infty[;\Omega_f \cup \Omega_c)$. Assume that the phase transition system in (19)-(27) satisfies the assumption **(H-2)** and that the initial condition $(\rho_r(x),q_r(x))$ satisfies the assumption **(H-1)** in $]0,+\infty[$, in the sense of Definitions 3.1 and 3.2. Then there exists $(\hat\rho_l,(\hat\rho_r,\hat q_r))$, a weak solution to (19) in the sense of Definition 4.1, such that*

1. *$\hat\rho_l(0,x) = \rho_l(x)$ for a.e. $x < 0$;*
2. *$(\hat\rho_r(0,x), \hat q_r(0,x)) = (\rho_r(x), q_r(x))$ for a.e. $x > 0$;*
3. *for a.e. $t > 0$*

$$\mathcal{RS}_{1,2}\left(\hat\rho_l(t,0-), (\hat\rho_r(t,0+), \hat q_r(t,0+))\right) = \left(\hat\rho_l(t,0-), (\hat\rho_r(t,0+), \hat q_r(t,0+))\right).$$

4.2.1 Wave-front tracking

Once we know the solution to Riemann problems, we are able to construct piecewise constant approximations via the wave-front tracking technique; see [4, 17] for the general theory and [12, 13] for the case of networks.

Definition 4.5 *Given $\varepsilon > 0$ and the Riemann solver $\mathcal{RS}_{1,2}$, we say that the map $\bar u_\varepsilon = (\bar\rho_{l,\varepsilon}, (\bar\rho_{r,\varepsilon}, \bar q_{r,\varepsilon}))$ is an ε-approximate wave-front tracking solution to (19)-(27) if the following conditions hold.*

1. *It holds that*

$$\begin{cases} \bar\rho_{l,\varepsilon} \in C([0,+\infty[; L^1_{loc}(]-\infty,0];[0,R])) \\ (\bar\rho_{r,\varepsilon}, \bar q_{r,\varepsilon}) \in C([0,+\infty[; L^1_{loc}([0,+\infty[;\Omega_f \cup \Omega_c)). \end{cases}$$

2. *$\bar\rho_{l,\varepsilon}(t,x)$ is an ε-approximate wave-front tracking solution to (1) on $x < 0$; see [4]. Moreover the jumps can be entropic shocks or rarefaction shocks and are indexed by $\mathcal{J}_l(t) = \mathcal{S}_l(t) \cup \mathcal{R}_l(t)$.*
3. *$(\bar\rho_{r,\varepsilon}, \bar q_{r,\varepsilon})$ is an ε-approximate wave-front tracking solution to the PT model on $x > 0$; see for example [6]. Moreover the jumps can be of the first family, of the second family, or of phase-transition waves. They are indexed by $\mathcal{J}_r(t) = 1_r(t) \cup 2_r(t) \cup \mathcal{PT}_r(t)$.*
4. *It holds that*

$$\begin{cases} \|\bar\rho_{l,\varepsilon}(0,\cdot) - \rho_l(\cdot)\|_{L^1(-\infty,0)} < \varepsilon \\ \mathit{Tot.Var.}\,\bar\rho_{l,\varepsilon}(0,\cdot) \le \mathit{Tot.Var.}\,\rho_l(\cdot) \\ \|(\bar\rho_{r,\varepsilon}(0,\cdot), \bar q_{r,\varepsilon}(0,\cdot)) - (\rho_r(\cdot), q_r(\cdot))\|_{L^1(0,+\infty)} < \varepsilon \\ \mathit{Tot.Var.}\,(\bar\rho_{r,\varepsilon}(0,\cdot), \bar q_{r,\varepsilon}(0,\cdot)) \le \mathit{Tot.Var.}\,(\rho_r(\cdot), q_r(\cdot)). \end{cases}$$

5. *For a.e. $t > 0$*

$$\mathcal{RS}_{1,2}\left(\bar\rho_{l,\varepsilon}(t,0-), (\bar\rho_{r,\varepsilon}, \bar q_{r,\varepsilon})(t,0+)\right) = \left(\bar\rho_{l,\varepsilon}(t,0-), (\bar\rho_{r,\varepsilon}, \bar q_{r,\varepsilon})(t,0+)\right).$$

Consider two sequences $\rho_{0,l,\nu}$ and $(\rho_{0,r,\nu}, q_{0,r,\nu})$ of piecewise constant functions defined respectively on $]-\infty,0]$ and $[0,+\infty[$ having a finite number of discontinuities and such that

1. $\lim_{\nu\to+\infty} \rho_{0,l,\nu} = \rho_l$ in $L^1_{loc}(]-\infty,0];[0,R])$;
2. $\lim_{\nu\to+\infty} (\rho_{0,r,\nu}, q_{0,r,\nu}) = (\rho_r, q_r)$ in $L^1_{loc}([0,+\infty[;\Omega_f \cup \Omega_c)$.

For every $\nu \in \mathbb{N} \setminus \{0\}$, we apply the following procedure. At time $t = 0$, we solve the Riemann problem at $x = 0$ (according to $\mathcal{RS}_{1,2}$) and all Riemann problems for $x < 0$ and $x > 0$. We approximate every rarefaction wave with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$ traveling with the Rankine-Hugoniot speed. Moreover, for $x < 0$, if σ is in the range of a rarefaction shock, then its speed is zero. We repeat the previous construction at every time at which interactions between waves or of waves with $x = 0$ happen.

Remark 4 *By slightly modifying the speed of waves, we may assume that, at every positive time t, at most one interaction happens. Moreover, at every interaction time, either two waves interact at $x \neq 0$ or a wave reaches the node $x = 0$.*

Remark 5 *For interactions at $x \neq 0$, we split rarefaction waves into rarefaction fans just at time $t = 0$. At $x = 0$, instead, we allow the formation of rarefaction fans at every positive time.*

Remark 6 *Since the velocity of each initial condition is greater than or equal to $\bar{v}$, then also the velocity v_c of every states in Ω_c, generated by the previous construction, is greater than or equal to $\bar{v}$.*

Given an ε-approximate wave-front tracking solution $\bar{u}_\varepsilon = (\bar{\rho}_{l,\varepsilon}, (\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon}))$, define, for a.e. $t > 0$, the following functionals (a detailed explanation is contained in Remark 7 and 8)

$$\mathrm{TV}^{\uparrow}_{f,1}(t) = \sum_{x \in \mathcal{I}_r(t)} \left[\varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x+)\right) - \varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x-)\right)\right]^+ \tag{28}$$

$$\mathrm{TV}^{\downarrow}_{f,1}(t) = \sum_{x \in \mathcal{I}_r(t)} \left[\varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x+)\right) - \varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x-)\right)\right]^- \tag{29}$$

$$\mathrm{TV}^{\uparrow}_{f,2}(t) = \sum_{x \in \mathcal{Z}_r(t)} \left[\varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x+)\right) - \varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x-)\right)\right]^+ \tag{30}$$

$$\mathrm{TV}^{\downarrow}_{f,2}(t) = \sum_{x \in \mathcal{Z}_r(t)} \left[\varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x+)\right) - \varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x-)\right)\right]^- \tag{31}$$

$$\mathrm{TV}^{\uparrow}_{f,PT}(t) = \sum_{x \in \mathcal{PT}_r(t)} \left[\varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x+)\right) - \varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x-)\right)\right]^+ \tag{32}$$

$$\mathrm{TV}^{\downarrow}_{f,PT}(t) = \sum_{x \in \mathcal{PT}_r(t)} \left[\varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x+)\right) - \varphi\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t, x-)\right)\right]^- \tag{33}$$

$$W_1(t) = \sum_{x \in \mathcal{I}_r(t)} \left|v_c\left(\bar{\rho}_{r,\varepsilon}(t, x+), \bar{q}_{r,\varepsilon}(t, x+)\right) - v_c\left(\bar{\rho}_{r,\varepsilon}(t, x-), \bar{q}_{r,\varepsilon}(t, x-)\right)\right| \tag{34}$$

$$W_2(t) = \sum_{x \in \mathcal{Z}_r(t)} \left|\omega\left(\bar{\rho}_{r,\varepsilon}(t, x+), \bar{q}_{r,\varepsilon}(t, x+)\right) - \omega\left(\bar{\rho}_{r,\varepsilon}(t, x-), \bar{q}_{r,\varepsilon}(t, x-)\right)\right| \tag{35}$$

$$\begin{aligned} W_{PT}(t) &= \sum_{x \in \mathcal{PT}_r(t)} \left[\omega\left(\sigma_-, q^-\sigma_-/R\right) - \omega\left(\left(\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon}\right)(t, x-)\right)\right] \\ &\quad + \sum_{x \in \mathcal{PT}_r(t)} \left[V - v_c\left(\left(\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon}\right)(t, x+)\right)\right] \end{aligned} \tag{36}$$

$$W_{1,2}(t) = \mathrm{Tot.Var.} f\left(\bar{\rho}_{l,\varepsilon}(t, \cdot)\right) + W_1(t) + W_2(t) + W_{PT}(t) \tag{37}$$

where the superscripts "+" and "−" denote the positive and negative part and $\omega : \Omega_f \cup \Omega_c \to \mathbb{R}$ is the continuous function defined by

$$\omega(\rho, q) = \begin{cases} \frac{q}{\rho} & \text{if } \frac{q}{\rho} \geq \frac{q^-}{R}, \\ V\rho - \sigma_- V + \frac{q^-}{R} & \text{otherwise.} \end{cases} \tag{38}$$

Note that the previous functionals may vary only at times $\bar{t}$ when two waves interact or a wave reaches the interface $x = 0$.

Remark 7 *The functional $TV^{\uparrow}_{f,1}(t)$ (resp. $TV^{\downarrow}_{f,1}(t)$) consists of the total variation of the flux at time t only due to waves of the first family with increasing flux (resp. with decreasing flux).*
The functional $TV^{\uparrow}_{f,2}(t)$ (resp. $TV^{\downarrow}_{f,2}(t)$) consists of the total variation of the flux at time t only due to waves of the second family with increasing flux (resp. with decreasing flux).
The functional $TV^{\uparrow}_{f,PT}(t)$ (resp. $TV^{\downarrow}_{f,PT}(t)$) consists of the total variation of the flux at time t only due to phase transition waves with increasing flux (resp. with decreasing flux).
Clearly $TV^{\uparrow}_{f,1}(t)+TV^{\downarrow}_{f,1}(t)+TV^{\uparrow}_{f,2}(t)+TV^{\downarrow}_{f,2}(t)+TV^{\uparrow}_{f,PT}(t)+TV^{\downarrow}_{f,PT}(t)$ gives the total variation of the flux at the time t for the phase transition model.

Remark 8 *The functional $W_{1,2}$ is composed by 4 terms. The first term is the total variation of the flux for the LWR model. The second and the third term measure, in the phase transition model, the strength of waves of first and second family. In particular, for waves of the second family, we introduced ω in order to avoid that a 2-wave connecting the state $(0,-1)$ has infinity strength. Finally, the last term measures the phase transition waves as a sum of a wave of the first family, connecting the right state with the point $(\sigma_-, q^-\sigma_-/R)$, and of a wave of the second family, connecting the left state with $(\sigma_-, q^-\sigma_-/R)$.*

4.2.2 Flux estimates for the LWR-PT model

We consider here interaction estimates for waves of the LWR-PT model. Let $\mathcal{RS}_{1,2}$ be the Riemann solver $\mathcal{RS}^1_{1,2}$, defined in Subsection 4.1.1, or the Riemann solver $\mathcal{RS}^2_{1,2}$, defined in Subsection 4.1.2. We have the following result in the case a wave interact at $x=0$ from the left.

Proposition 4.3 *Let $\rho_l^- \in [0,R]$, $(\rho_r^-, q_r^-) \in \Omega_f \cup \Omega_c$ be such that*

$$\mathcal{RS}_{1,2}\left(\rho_l^-, (\rho_r^-, q_r^-)\right) = \left(\rho_l^-, (\rho_r^-, q_r^-)\right). \tag{39}$$

Assume that a wave (ρ_l, ρ_l^-) interacts with the interface $x=0$ at a time $\bar t > 0$. Then

$$Tot.Var.^f_{1,2}(\bar t-) = Tot.Var.^f_{1,2}(\bar t+). \tag{40}$$

Proof. Define $\rho_l^+ \in [0,R]$ and $(\rho_r^+, q_r^+) \in \Omega_f \cup \Omega_c$ by the relation

$$\left(\rho_l^+, (\rho_r^+, q_r^+)\right) = \mathcal{RS}_{1,2}\left(\rho_l, (\rho_r^-, q_r^-)\right);$$

see Figure 4. Note that $f(\rho_l^-) = \varphi(\rho_r^-, q_r^-)$ and $f(\rho_l^+) = \varphi(\rho_r^+, q_r^+)$. We clearly have that

$$\begin{aligned}
\text{Tot.Var.}^f_{1,2}(\bar t+) - \text{Tot.Var.}^f_{1,2}(\bar t-) &= \left|f(\rho_l) - f(\rho_l^+)\right| - \left|f(\rho_l) - f(\rho_l^-)\right| \\
&\quad + \left|\varphi(\rho_r^-, q_r^-) - \varphi(\rho_r^+, q_r^+)\right| \\
&= \left|f(\rho_l) - f(\rho_l^+)\right| - \left|f(\rho_l) - f(\rho_l^-)\right| \\
&\quad + \left|f(\rho_l^-) - f(\rho_l^+)\right|.
\end{aligned}$$

Moreover, since the wave (ρ_l, ρ_l^-) have non-negative speed, then $\rho_l \le \frac{R}{2}$ and, by (3) and by Proposition 2.1, we deduce that $f(\rho_l) \ge f(\rho_l^+)$. Therefore

$$\begin{aligned}
\text{Tot.Var.}^f_{1,2}(\bar t+) - \text{Tot.Var.}^f_{1,2}(\bar t-) &= f(\rho_l) - f(\rho_l^+) - \left|f(\rho_l) - f(\rho_l^-)\right| \\
&\quad + \left|f(\rho_l^-) - f(\rho_l^+)\right|.
\end{aligned}$$

We have two different possibilities.

1. $f(\rho_l^+) \ge f(\rho_l^-)$. In this case, since $f(\rho_l) \ge f(\rho_l^+)$, we easily deduce that (40) holds.
2. $f(\rho_l^+) < f(\rho_l^-)$. In this case we claim that $\rho_l = \rho_l^+$. Indeed, since $\rho_l \le \frac{R}{2}$, then, by (3), $\rho_l^+ = \rho_l$ or $\rho_l^+ > R - \rho_l$. Assume, by contradiction, that $\rho_l^+ > R - \rho_l$; then the set $\Gamma_{\rho_l,(\rho_r^-, q_r^-)}$, defined in (25), is bigger than or equal to $\Gamma_{\rho_l^-,(\rho_r^-, q_r^-)}$, but this contradicts $f(\rho_l^+) < f(\rho_l^-)$. Therefore $\rho_l = \rho_l^+$ and so (40) holds.

The proof is so finished. □

In the case of an interaction from the right we have the following proposition.

PSfrag replacements

$\bar{t}$

$x{=}0$

ρ_l^-

ρ_l

ρ_l^+

(ρ_r^-, q_r^-)

(ρ_r^+, q_r^+)

Figure 4: An interaction from the left for the LWR-PT model.

Proposition 4.4 *Let $\rho_l^- \in [0,R]$, $(\rho_r^-, q_r^-) \in \Omega_f \cup \Omega_c$ be such that*

$$\mathcal{RS}_{1,2}\left(\rho_l^-, (\rho_r^-, q_r^-)\right) = \left(\rho_l^-, (\rho_r^-, q_r^-)\right). \tag{41}$$

Assume that a wave $((\rho_r^-, q_r^-), (\rho_r, q_r))$ interacts with the interface $x = 0$ at a time $\bar{t} > 0$. Then

$$Tot.Var.^f_{1,2}(\bar{t}-) = Tot.Var.^f_{1,2}(\bar{t}+). \tag{42}$$

Proof. Define $\rho_l^+ \in [0,R]$ and $(\rho_r^+, q_r^+) \in \Omega_f \cup \Omega_c$ by the relation

$$\left(\rho_l^+, (\rho_r^+, q_r^+)\right) = \mathcal{RS}_{1,2}\left(\rho_l^-, (\rho_r, q_r)\right);$$

see Figure 5. Note that $f(\rho_l^-) = \varphi(\rho_r^-, q_r^-)$ and $f(\rho_l^+) = \varphi(\rho_r^+, q_r^+)$. Since the velocity of the interacting wave is different from 0, then $\varphi(\rho_r^-, q_r^-) \neq \varphi(\rho_r, q_r)$. We clearly have that

$$\begin{aligned} \text{Tot.Var.}^f_{1,2}(\bar{t}+) - \text{Tot.Var.}^f_{1,2}(\bar{t}-) &= \left|f(\rho_l^-) - f(\rho_l^+)\right| \\ &- \left|\varphi(\rho_r, q_r) - \varphi(\rho_r^-, q_r^-)\right| + \left|\varphi(\rho_r, q_r) - \varphi(\rho_r^+, q_r^+)\right| \\ &= \left|\varphi(\rho_r^-, q_r^-) - \varphi(\rho_r^+, q_r^+)\right| - \left|\varphi(\rho_r, q_r) - \varphi(\rho_r^-, q_r^-)\right| \\ &+ \left|\varphi(\rho_r, q_r) - \varphi(\rho_r^+, q_r^+)\right|. \end{aligned} \tag{43}$$

There are the following possibilities.

1. (ρ_r^-, q_r^-) provides a constraint for the equilibrium $(\rho_l^-, (\rho_r^-, q_r^-))$. In this case $\frac{q_r^-}{\rho_r^-} = \frac{q^+}{R}$ and $(\rho_r^-, q_r^-) \in \Omega_c$ and, since the interacting wave has strictly negative speed, we also deduce that $(\rho_r, q_r) \in \Omega_c$, $\frac{q_r}{\rho_r} = \frac{q^+}{R}$, $\varphi(\rho_r, q_r) \neq \varphi(\rho_r^-, q_r^-)$ and the interacting wave is a wave of the first family.
   If $\varphi(\rho_r, q_r) < \varphi(\rho_r^-, q_r^-)$, then we conclude that $(\rho_r^+, q_r^+) = (\rho_r, q_r)$ and so (43) is equal to 0.
   If $\varphi(\rho_r, q_r) > \varphi(\rho_r^-, q_r^-)$, then the set $\Gamma_{\rho_l^-, (\rho_r^-, q_r^-)}$ is contained in $\Gamma_{\rho_l^-, (\rho_r, q_r)}$ (also the equality is possible). Therefore we deduce that $\varphi(\rho_r^-, q_r^-) \leq \varphi(\rho_r^+, q_r^+) \leq \varphi(\rho_r, q_r)$ and so (43) is equal to 0.

2. (ρ_r^-, q_r^-) is not a constraint for the equilibrium $(\rho_l^-, (\rho_r^-, q_r^-))$. In this case the wave $((\rho_r^-, q_r^-), (\rho_r, q_r))$ is either a wave of the first family or a phase transition wave.
   If the wave $((\rho_r^-, q_r^-), (\rho_r, q_r))$ is of the first family and
   $$\sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r, q_r)} \varphi(\rho, q) \geq \varphi(\rho_r^-, q_r^-),$$
   then the fluxes for the solutions before and after the interaction coincide since (ρ_r^-, q_r^-) is not a constraint for the equilibrium $(\rho_l^-, (\rho_r^-, q_r^-))$, and so $\rho_l^+ = \rho_l^-$ and $\varphi(\rho_r^+, q_r^+) = \varphi(\rho_r^-, q_r^-)$. Hence (43) is equal to 0.

   If the wave $((\rho_r^-, q_r^-), (\rho_r, q_r))$ is of the first family and
   $$\sup_{(\rho,q) \in \mathcal{T}_r^2(\rho_r, q_r)} \varphi(\rho, q) < \varphi(\rho_r^-, q_r^-),$$
   then $\varphi(\rho_r, q_r) \leq \varphi(\rho_r^+, q_r^+) < \varphi(\rho_r^-, q_r^-)$; hence (43) is equal to 0.

   If $((\rho_r^-, q_r^-), (\rho_r, q_r))$ is a phase transition wave, then $(\rho_r^-, q_r^-) \in \Omega_f$, $(\rho_r, q_r) \in \Omega_c$, $\varphi(\rho_r, q_r) < \varphi(\rho_r^-, q_r^-)$ and $(\rho_r, q_r) = \psi_2^-(\rho_r, q_r)$, where the function ψ_2^- is defined in (11). Therefore $\Gamma_{\rho_l^-, (\rho_r^-, q_r^-)} \geq \Gamma_{\rho_l^-, (\rho_r, q_r)}$, which implies $\varphi(\rho_r^+, q_r^+) \leq \varphi(\rho_r^-, q_r^-)$ and $\varphi(\rho_r, q_r) \leq \varphi(\rho_r^+, q_r^+)$; hence (43) is equal to 0.

PSfrag replacements

$\bar{t}$

$x=0$

ρ_l^-

ρ_l^+

(ρ_r, q_r)

(ρ_r^-, q_r^-)

(ρ_r^+, q_r^+)

Figure 5: An interaction from the right for the LWR-PT model.

The proof is so finished. □

For an interaction of two waves at a point $x < 0$, since we have a scalar conservation law, the following result holds; see [12].

Lemma 4.2 *Assume that the waves (ρ^l, ρ^m) and (ρ^m, ρ^r) interact at the point $(\bar{t}, \bar{x})$ with $\bar{t} > 0$ and $\bar{x} < 0$. Then*

$$Tot.\,Var.^f_{1,2}(\bar{t}+) \leq Tot.\,Var.^f_{1,2}(\bar{t}-). \tag{44}$$

For an interaction of two waves at a point $x > 0$, we have the following results.

Lemma 4.3 *Assume that the waves $((\rho^l, q^l), (\rho^m, q^m))$ and $((\rho^m, q^m), (\rho^r, q^r))$ interact at the point $(\bar{t}, \bar{x})$ with $\bar{t} > 0$ and $\bar{x} > 0$. Then $((\rho^m, q^m), (\rho^r, q^r))$ is a wave of the first family or a phase-transition wave.*

Proof. Suppose by contradiction that the wave $((\rho^m, q^m), (\rho^r, q^r))$ is not a wave of the first family or a phase-transition wave; thus we have the following possibilities.

1. $((\rho^m, q^m), (\rho^r, q^r))$ is completely contained in Ω_f. In this case its speed is V, the biggest speed allowed. Therefore it can not interact with the wave $((\rho^l, q^l), (\rho^m, q^m))$.

2. $((\rho^m, q^m), (\rho^r, q^r))$ is a wave of the second family in Ω_c and so it has positive speed. By the previous case, we can assume that $((\rho^m, q^m), (\rho^r, q^r))$ is completely contained in $\Omega_c \setminus \Omega_f$. Clearly, $((\rho^l, q^l), (\rho^m, q^m))$ is not a wave of the first family, otherwise its speed is negative and so the interaction does not happen. Therefore $((\rho^l, q^l), (\rho^m, q^m))$ could be a wave of the second family or a phase-transition wave.
   If $((\rho^l, q^l), (\rho^m, q^m))$ is a wave of the second family, then it has the same speed of $((\rho^m, q^m), (\rho^r, q^r))$, since the second characteristic field is linearly degenerate. Hence no interaction happens and this case is not possible.
   If $((\rho^l, q^l), (\rho^m, q^m))$ is a phase-transition wave, then $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ and its speed is less than the speed of $((\rho^m, q^m), (\rho^r, q^r))$. Hence this case is not possible.

The proof is so completed. □

Proposition 4.5 *Suppose that the phase transition system in (19)-(27) satisfies the assumption **(H-2)** and the initial condition (ρ_r, q_r) satisfies the assumption **(H-1)** in $]0, +\infty[$ in the sense of Definitions 3.1 and 3.2. Assume that the wave $((\rho^l, q^l), (\rho^m, q^m))$ interacts with the wave $((\rho^m, q^m), (\rho^r, q^r))$ at the point $(\bar{t}, \bar{x})$ with $\bar{t} > 0$ and $\bar{x} > 0$. There exists a constant $K > 0$ with the following properties. If $((\rho^l, q^l), (\rho^m, q^m))$ is a wave of the second family with decreasing flux and $((\rho^m, q^m), (\rho^r, q^r))$ is a wave of the first family with increasing flux, then $\Delta TV^{\uparrow}_{f,1}(\bar{t}) = \Delta TV^{\downarrow}_{f,2}(\bar{t}) > 0$ and*

$$\Delta\, Tot.\,Var.^f_{1,2}(\bar{t}+) \leq K \min\left\{\left|\varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)\right|, \left|\varphi(\rho^l, q^l) - \varphi(\rho^m, q^m)\right|\right\}.$$

In all the remaining cases, we have

$$Tot.\,Var.^f_{1,2}(\bar{t}+) \leq Tot.\,Var.^f_{1,2}(\bar{t}-).$$

PROOF. By Lemma 4.3, the wave $((\rho^m, q^m), (\rho^r, q^r))$ is either a wave of the first family in Ω_c or a phase-transition wave.

First case. $((\rho^m, q^m), (\rho^r, q^r))$ is a phase-transition wave. This implies that $(\rho^m, q^m) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) \in \Omega_c \setminus \Omega_f$. Since (ρ^m, q^m) is the right state of the wave $((\rho^l, q^l), (\rho^m, q^m))$, then we deduce that $(\rho^l, q^l) \in \Omega_f$, i.e. $((\rho^l, q^l), (\rho^m, q^m))$ is a contact discontinuity wave contained in Ω_f.

Assume first that $\psi_2^-(\rho^r, q^r) \neq (\rho^r, q^r)$. In this situation necessarily the state (ρ^m, q^m) is $(0, -1)$. If $(\rho^l, q^l) \in \Omega_f \cap \Omega_c$, then the solution to the Riemann problem with data $((\rho^l, q^l), (\rho^r, q^r))$ is solved by a wave of the first family and possibly by a wave of the second family, both contained in Ω_c. Denote by $(\tilde\rho, \tilde q)$ the intermediate state generated by this Riemann problem; hence

$$\begin{aligned}
\Delta\text{Tot.Var.}_{1,2}^f(\bar t) &= \left|\varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q)\right| + |\varphi(\tilde\rho, \tilde q) - \varphi(\rho^r, q^r)| \\
&\quad - \left|\varphi(\rho^l, q^l) - \varphi(\rho^m, q^m)\right| - |\varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)| \\
&= \varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q) + |\varphi(\tilde\rho, \tilde q) - \varphi(\rho^r, q^r)| \\
&\quad -\varphi(\rho^l, q^l) - \varphi(\rho^r, q^r) \\
&= |\varphi(\tilde\rho, \tilde q) - \varphi(\rho^r, q^r)| - \varphi(\rho^r, q^r) - \varphi(\tilde\rho, \tilde q)
\end{aligned}$$

and the conclusion follows by the triangular inequality. If $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, then the solution to the Riemann problem with data $((\rho^l, q^l), (\rho^r, q^r))$ is solved by a phase transition wave (possibly followed by a wave of the first family) and by a wave of the second family. Define $(\tilde\rho, \tilde q) = \psi_2^-(\rho^r, q^r)$; hence

$$\begin{aligned}
\Delta\text{Tot.Var.}_{1,2}^f(\bar t) &= \left|\varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q)\right| + |\varphi(\tilde\rho, \tilde q) - \varphi(\rho^r, q^r)| \\
&\quad - \left|\varphi(\rho^l, q^l) - \varphi(\rho^m, q^m)\right| - |\varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)| \\
&= \left|\varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q)\right| + \varphi(\rho^r, q^r) - \varphi(\tilde\rho, \tilde q) \\
&\quad -\varphi(\rho^l, q^l) - \varphi(\rho^r, q^r) \\
&= \left|\varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q)\right| - \varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q)
\end{aligned}$$

and the conclusion follows by the triangular inequality.

Assume now that $\psi_2^-(\rho^r, q^r) = (\rho^r, q^r)$. If $(\rho^l, q^l) \in \Omega_f \cap \Omega_c$, then the solution to the Riemann problem with data $((\rho^l, q^l), (\rho^r, q^r))$ is solved by a wave of the first family and a wave of the second family, both contained in Ω_c; hence $\Delta\text{Tot.Var.}_{1,2}^f(\bar t)$ is given by

$$\left|\varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)\right| - \left|\varphi(\rho^l, q^l) - \varphi(\rho^m, q^m)\right| - |\varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)|$$

and the conclusion follows by the triangular inequality. If $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, then the solution to the Riemann problem with data $((\rho^l, q^l), (\rho^r, q^r))$ is solved by a phase transition wave or by a phase transition wave coupled with a wave of the first family. In both cases we conclude by the triangular inequality, since the waves in the second case are monotonic with respect to the flux φ.

Second case. $((\rho^m, q^m), (\rho^r, q^r))$ is a wave of the first family. We have the following possibilities.

1. $((\rho^l, q^l), (\rho^m, q^m))$ is a contact discontinuity wave of the second family with $(\rho^l, q^l) \in \Omega_c$. The Riemann problem with data $((\rho^l, q^l), (\rho^r, q^r))$ is solved by a wave of the first family and by a wave of the second family, both contained in Ω_c. Denote by $(\tilde\rho, \tilde q)$ the intermediate state of this Riemann problem.
   If $\varphi(\rho^l, q^l) < \varphi(\rho^m, q^m) < \varphi(\rho^r, q^r)$ or $\varphi(\rho^r, q^r) < \varphi(\rho^m, q^m) < \varphi(\rho^l, q^l)$, then

   $$\min\left\{\varphi(\rho^l, q^l), \varphi(\rho^r, q^r)\right\} < \varphi(\tilde\rho, \tilde q) < \max\left\{\varphi(\rho^l, q^l), \varphi(\rho^r, q^r)\right\}$$

   and so $\text{Tot.Var.}_{1,2}^f(\bar t+) = \text{Tot.Var.}_{1,2}^f(\bar t-)$.
   If (ρ^m, q^m) satisfies $\varphi(\rho^m, q^m) < \min\left\{\varphi(\rho^l, q^l), \varphi(\rho^r, q^r)\right\}$ then, by **(H-2)**, $\Delta\text{TV}_{f,2}^{\downarrow}(\bar t) = \Delta\text{TV}_{f,1}^{\uparrow}(\bar t) > 0$. More precisely, by Proposition 3.1, we deduce the existence of a constant $C > 1$, which depends only by $\bar v$ of condition **(H-1)**, such that

   $$\begin{aligned}
   \Delta\text{TV}_{f,2}^{\downarrow}(\bar t) &= \Delta\text{TV}_{f,1}^{\uparrow}(\bar t) = \varphi(\tilde\rho, \tilde q) - \varphi(\rho^l, q^l) - \varphi(\rho^r, q^r) + \varphi(\rho^m, q^m) \\
   &\le C\left(\varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)\right) - \varphi(\rho^r, q^r) + \varphi(\rho^m, q^m) = [C-1]\left(\varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)\right)
   \end{aligned}$$

   and

   $$\begin{aligned}
   \Delta\text{TV}_{f,2}^{\downarrow}(\bar t) &= \Delta\text{TV}_{f,1}^{\uparrow}(\bar t) = \varphi(\tilde\rho, \tilde q) - \varphi(\rho^r, q^r) - \varphi(\rho^l, q^l) + \varphi(\rho^m, q^m) \\
   &\le C\left(\varphi(\rho^l, q^l) - \varphi(\rho^m, q^m)\right) - \varphi(\rho^l, q^l) + \varphi(\rho^m, q^m) = [C-1]\left(\varphi(\rho^l, q^l) - \varphi(\rho^m, q^m)\right);
   \end{aligned}$$

Hence

$$\Delta \text{Tot.Var.}^{f}_{1,2}(\bar{t}+) \le 2\,(C-1)\,\min\left\{\left(\varphi(\rho^r,q^r)-\varphi(\rho^m,q^m)\right),\left(\varphi(\rho^l,q^l)-\varphi(\rho^m,q^m)\right)\right\}.$$

If $\varphi(\rho^m,q^m) > \max\left\{\varphi(\rho^l,q^l),\varphi(\rho^r,q^r)\right\}$, then, by **(H-2)**, $\Delta \mathrm{TV}^{\downarrow}_{f,1}(\bar{t}) = \Delta \mathrm{TV}^{\uparrow}_{f,2}(\bar{t}) < 0$ and so we obtain that $\Delta \text{Tot.Var.}^{f}_{1,2}(\bar{t}) < 0$.

2. $((\rho^l,q^l),(\rho^m,q^m))$ is a contact discontinuity wave of the second family with $(\rho^l,q^l) \in \Omega_f \setminus \Omega_c$. Note that $(\rho^m,q^m) \in \Omega_c \cap \Omega_f$ and so
$$\varphi(\rho^m,q^m) > \max\left\{\varphi(\rho^l,q^l),\varphi(\rho^r,q^r)\right\}.$$
Define $(\bar{\rho},\bar{q}) = \psi_2^-(\rho^r,q^r)$ and $(\tilde{\rho},\tilde{q}) = \psi_2^-(\rho^m,q^m)$, where ψ_2^- is defined in (11). The Riemann problem with data $((\rho^l,q^l),(\rho^r,q^r))$ is solved by a phase transition wave, possibly coupled with a wave of the first family, connecting (ρ^l,q^l) to $(\bar{\rho},\bar{q})$, and by a wave of the second family, connecting $(\bar{\rho},\bar{q})$ to (ρ^r,q^r). Notice that, if a wave of the first family is present, then the flux φ is monotone in the first two waves. Hence
$$\begin{aligned}\Delta \text{Tot.Var.}^{f}_{1,2}(\bar{t}) &= \left|\varphi(\rho^l,q^l)-\varphi(\bar{\rho},\bar{q})\right| + \left|\varphi(\rho^r,q^r)-\varphi(\bar{\rho},\bar{q})\right| \\ &\quad - \left|\varphi(\rho^l,q^l)-\varphi(\rho^m,q^m)\right| - \left|\varphi(\rho^m,q^m)-\varphi(\rho^r,q^r)\right| \\ &= \left|\varphi(\rho^l,q^l)-\varphi(\bar{\rho},\bar{q})\right| + \varphi(\rho^r,q^r)-\varphi(\bar{\rho},\bar{q}) \\ &\quad -\varphi(\rho^m,q^m)+\varphi(\rho^l,q^l)-\varphi(\rho^m,q^m)+\varphi(\rho^r,q^r) \\ &= \left|\varphi(\rho^l,q^l)-\varphi(\bar{\rho},\bar{q})\right| + 2\varphi(\rho^r,q^r)-2\varphi(\rho^m,q^m) \\ &\quad +\varphi(\rho^l,q^l)-\varphi(\bar{\rho},\bar{q}).\end{aligned}$$
If $\varphi(\rho^l,q^l) \le \varphi(\bar{\rho},\bar{q})$, then $\Delta \text{Tot.Var.}^{f}_{1,2}(\bar{t}) = 2(\varphi(\rho^r,q^r)-\varphi(\rho^m,q^m)) < 0$ and the conclusion follows. If $\varphi(\rho^l,q^l) > \varphi(\bar{\rho},\bar{q})$, then
$$\begin{aligned}\Delta \text{Tot.Var.}^{f}_{1,2}(\bar{t}) &= 2\left[\varphi(\rho^l,q^l)-\varphi(\bar{\rho},\bar{q})+\varphi(\rho^r,q^r)-\varphi(\rho^m,q^m)\right] \\ &\le 2\left[\varphi\left(\sigma_-,\frac{q^-\sigma_-}{R}\right)-\varphi(\bar{\rho},\bar{q})+\varphi(\rho^r,q^r)-\varphi(\rho^m,q^m)\right],\end{aligned}$$
which is lower than or equal to 0, by **(H-2)**.

3. $((\rho^l,q^l),(\rho^m,q^m))$ is a phase transition wave. In this case we deduce that $(\rho^l,q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^m,q^m) \in \Omega_c \setminus \Omega_f$ and $(\rho^m,q^m) = \psi_2^-(\rho^m,q^m)$. If $(\rho^r,q^r) \in \Omega_f$, then the solution to the Riemann problem with data $((\rho^l,q^l),(\rho^r,q^r))$ consists in a wave of the second family; therefore we have that $\Delta \text{Tot.Var.}^{f}_{1,2}(\bar{t})$ is equal to
$$\left|\varphi(\rho^l,q^l)-\varphi(\rho^r,q^r)\right| - \left|\varphi(\rho^l,q^l)-\varphi(\rho^m,q^m)\right| - \left|\varphi(\rho^m,q^m)-\varphi(\rho^r,q^r)\right|,$$
which is lower than or equal to 0 by the triangular inequality. In the other case, the solution to the Riemann problem with initial data (ρ^l,q^l) and (ρ^r,q^r) is composed by a phase transition wave, possibly coupled with a wave of the first family; hence $\Delta \text{Tot.Var.}^{f}_{1,2}(\bar{t})$ is equal to
$$\left|\varphi(\rho^l,q^l)-\varphi(\rho^r,q^r)\right| - \left|\varphi(\rho^l,q^l)-\varphi(\rho^m,q^m)\right| - \left|\varphi(\rho^m,q^m)-\varphi(\rho^r,q^r)\right|$$
and the conclusion follows by the triangular inequality.

The proof is so completed. □

Remark 9 *By the proof of Proposition 4.5, there is only one type of interaction producing an increment of the functional Tot.Var.$^{f}_{1,2}$. More precisely, the increment of Tot.Var.$^{f}_{1,2}$ happens when a wave of the second family connecting $(\rho^l,q^l) \in \Omega_c$ and $(\rho^m,q^m) \in \Omega_c$ (with $\rho^m < \rho^l$) interacts with a wave of the first family connecting $(\rho^m,q^m) \in \Omega_c$ and $(\rho^r,q^r) \in \Omega_c$ (with $\rho^r < \rho^m$). In this situation, the wave of the second family has decreasing flux, while the wave of the first family has increasing flux; see Figure 6.*

Lemma 4.4 *Assume that a wave $((\rho^l,q^l),(\rho^r,q^r))$, generated at time $t_1 > 0$ in the PT model from J, interacts with a wave $((\tilde{\rho},\tilde{q}),(\rho^l,q^l))$ from the left at time $t_2 > t_1$. Suppose also that the wave $((\rho^l,q^l),(\rho^r,q^r))$ does not interact with other waves in the time interval (t_1,t_2). Then Tot.Var.$^{f}_{1,2}(t_2+) \le$ Tot.Var.$^{f}_{1,2}(t_2-)$.*

PSfrag replacements

ρ

0

ρv

(ρ^l, q^l)

(ρ^m, q^m)

(ρ^r, q^r)

Figure 6: The interaction producing an increment of the total variation

PROOF. Since the wave $((\rho^l, q^l), (\rho^r, q^r))$ has positive speed, then it can be a wave of the second family or a phase-transition wave. We claim that it is a phase transition wave. Assume, by contradiction, that it is a wave of the second family. We have two possibilities.

1. $(\rho^l, q^l), (\rho^r, q^r) \in \Omega_f$. In this case the speed of the wave is V and so no other wave from the left can reach it, since V is the maximum possible speed.

2. $(\rho^l, q^l), (\rho^r, q^r) \in \Omega_c$. If $((\tilde\rho, \tilde q), (\rho^l, q^l))$ is a wave of the second family, then it has the same speed of $((\rho^l, q^l), (\rho^r, q^r))$; hence it is not possible.
   If $((\tilde\rho, \tilde q), (\rho^l, q^l))$ is a phase transition wave, then its speed is lower than or equal to the speed of $((\rho^l, q^l), (\rho^r, q^r))$; hence it is not possible.

Since $((\rho^l, q^l), (\rho^r, q^r))$ is a phase transition wave with positive speed, then $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) \in \Omega_c \setminus \Omega_f$ and $\varphi(\rho^l, q^l) < \varphi(\rho^r, q^r)$. Moreover we deduce that also $(\tilde\rho, \tilde q)$ belongs to Ω_f. Denote by $(\rho^m, q^m) = \psi_2^-(\rho^r, q^r)$. Note that $\varphi(\rho^l, q^l) < \varphi(\rho^m, q^m)$ and that $(\rho^m, q^m) \neq (\rho^r, q^r)$ implies that $(\rho^l, q^l) = (0, -1)$. We have various possibilities.

1. $\tilde\rho < \sigma_-$. In this case the interaction at time t_2 produces a phase transition wave, possibly coupled with a wave of first family and with a wave of the second family. Therefore

$$\begin{aligned}\Delta\mathrm{Tot.Var.}^f_{1,2}(t_2) &= |\varphi(\tilde\rho, \tilde q) - \varphi(\rho^m, q^m)| + |\varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)| \\ &\quad - \left|\varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)\right| - \left|\varphi(\tilde\rho, \tilde q) - \varphi(\rho^l, q^l)\right| \\ &= |\varphi(\tilde\rho, \tilde q) - \varphi(\rho^m, q^m)| - \varphi(\rho^m, q^m) \\ &\quad + \varphi(\rho^l, q^l) - \left|\varphi(\tilde\rho, \tilde q) - \varphi(\rho^l, q^l)\right| \leq 0\end{aligned}$$

   by triangular inequality, since $\varphi(\rho^l, q^l) < \varphi(\rho^m, q^m)$.

2. $\tilde\rho = \sigma_-$. In this case the interaction at time t_2 produces a wave of the first family possibly coupled with a wave of the second family. Therefore

$$\begin{aligned}\Delta\mathrm{Tot.Var.}^f_{1,2}(t_2) &= |\varphi(\tilde\rho, \tilde q) - \varphi(\rho^m, q^m)| + |\varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)| \\ &\quad - \left|\varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)\right| - \left|\varphi(\tilde\rho, \tilde q) - \varphi(\rho^l, q^l)\right| \\ &= 2\varphi(\rho^l, q^l) - 2\varphi(\rho^m, q^m) \leq 0\end{aligned}$$

   since $\varphi(\rho^l, q^l) < \varphi(\rho^m, q^m)$.

3. $\tilde\rho > \sigma_-$. In this case the interaction at time t_2 produces a wave of the first family possibly coupled with a wave of the second family. Therefore, denoting with $(\tilde\rho^m, \tilde q^m)$ the intermediate state of the solution connecting $(\tilde\rho, \tilde q)$ with $\varphi(\rho^r, q^r)$,

$$\begin{aligned}\Delta\mathrm{Tot.Var.}^f_{1,2}(t_2) &= |\varphi(\tilde\rho, \tilde q) - \varphi(\tilde\rho^m, \tilde q^m)| + |\varphi(\tilde\rho^m, \tilde q^m) - \varphi(\rho^r, q^r)| \\ &\quad - \left|\varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)\right| - \left|\varphi(\tilde\rho, \tilde q) - \varphi(\rho^l, q^l)\right| \\ &= -\varphi(\tilde\rho^m, \tilde q^m) + |\varphi(\tilde\rho^m, \tilde q^m) - \varphi(\rho^r, q^r)| \\ &\quad - \varphi(\rho^r, q^r) + 2\varphi(\rho^l, q^l) \\ &\leq \left|\varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)\right| - \varphi(\rho^r, q^r) + \varphi(\rho^l, q^l) = 0,\end{aligned}$$

by triangular inequality.

The proof is completed. $\square$

Lemma 4.5 *Let $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ be a wave interacting with $x = 0$ at a time $\bar{t} > 0$. Suppose that*

- *a wave $((\rho^m, q^m), (\rho^r, q^r))$ of the second family with decreasing flux is produced at time $\bar{t}$*

or

- *a wave at time $\bar{t}$ is produced in the LWR model, which comes back to $x = 0$ producing a wave of the second family with decreasing flux at time $\tilde{t} > \bar{t}$.*

Then $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ is either a wave of the first family with decreasing flux or a phase transition wave with decreasing flux. Moreover the sum of the flux variation produced at $x = 0$ at time $\bar{t}$ and $\tilde{t}$ is less than or equal to

$$\left|\varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)\right|,$$

which is the flux variation of the interacting wave.

Proof. Since the wave $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ has negative speed, then it can be a wave of the first family with increasing or decreasing flux or a phase transition wave with decreasing flux.
Assume first that $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ is a wave of the first family with increasing flux. We have the following possibilities.

1. $\tilde{t} = \bar{t}$. In this case the reflected 2-wave should be with increasing flux; hence it is not possible.
2. $\tilde{t} > \bar{t}$ and no wave is produced at time $\bar{t}$ at the right of the interface. Then in the LWR model a rarefaction wave with increasing flux is produced. By Proposition 4.4, the two waves have the same flux variation. The rarefaction can come back to the interface (transforming itself in a shock wave with increasing flux) and so the wave $((\rho^m, q^m), (\rho^r, q^r))$ should have increasing flux. It is not possible.
3. $\tilde{t} > \bar{t}$ and a wave is produced a time $\bar{t}$ at the right of the interface (wave of the second family with increasing flux) and a wave (rarefaction with increasing flux) is generated at the same time in the LWR model. By Proposition 4.4, the wave in the LWR model has flux variation less than that of the interacting wave. The rarefaction can come back to the interface (transforming itself in a shock wave with increasing flux), and so the wave of the second family, generated at $\tilde{t}$ should have increasing flux. It is not possible.

Assume now that $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ is a wave of the first family with decreasing flux. We have three possibilities.

1. $\tilde{t} = \bar{t}$. In this case the reflected 2-wave should be with decreasing flux and it has flux variation less than or equal to that of the interacting wave, by Proposition 4.4.
2. $\tilde{t} > \bar{t}$ and no wave is produced at time $\bar{t}$ at the right of the interface. Then in the LWR model a shock with decreasing flux is produced. By [12, Lemma A.1.6], when it comes back to $x = 0$, it cancels its flux variation and so the conclusion follows.
3. $\tilde{t} > \bar{t}$ and a wave is produced a time $\bar{t}$ at the right of the interface (wave of the second family with decreasing flux) and a wave (shock with decreasing flux) is generated at the same time in the LWR model. We conclude as in the previous case.

Assume finally that $\big((\rho^l, q^l), (\rho^r, q^r)\big)$ is a phase transition wave with decreasing flux. This situation is completely similar to the previous one. The proof is so completed. $\square$

The next proposition deals with the interactions occurring inside the phase transition model ($x > 0$) and involving a phase transition wave. There are ten types of such interactions classified according to the type of interacting waves and those appearing after the interaction. We use the symbols 1, 2 and PT to indicate respectively a wave of first, second family and a phase transition.

Proposition 4.6 *Let us consider the PT model (7)-(8) and assume it satisfies the assumption **(H-2)**, in the sense of Definition 3.2. Consider an interaction involving a phase transition at time $\bar{t}$. The following possibilities hold.*

1. *If the interaction is* **2-PT/PT**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$, $\Delta TV_{f,PT}^{\uparrow}(\bar{t}) \le -\Delta TV_{f,2}^{\uparrow}(\bar{t})$, $\Delta TV_{f,PT}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,2}^{\downarrow}(\bar{t})$.

2. *If the interaction is* **2-PT/PT-1**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$ *and* $\Delta TV_{f,PT}^{\downarrow}(\bar{t}) + \Delta TV_{f,1}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,2}^{\downarrow}(\bar{t})$.

3. *If the interaction is* **2-PT/1-2**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$, $\Delta TV_{f,2}^{\downarrow}(\bar{t}) < 0$ *and* $\Delta TV_{f,2}^{\downarrow}(\bar{t}) + \Delta TV_{f,1}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,PT}^{\downarrow}(\bar{t})$.

4. *If the interaction is* **PT-1/2**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$ *and* $\Delta TV_{f,2}^{\uparrow}(\bar{t}) \le -\Delta TV_{f,1}^{\uparrow}(\bar{t}) - \Delta TV_{f,PT}^{\uparrow}(\bar{t})$.

5. *If the interaction is* **PT-1/PT**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$, $\Delta TV_{f,PT}^{\uparrow}(\bar{t}) \le -\Delta TV_{f,1}^{\uparrow}(\bar{t})$ *and* $\Delta TV_{f,PT}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,1}^{\downarrow}(\bar{t})$.

6. *If the interaction is* **PT-1/PT-1**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$, $\Delta TV_{f,1}^{\downarrow}(\bar{t}) < 0$ *and* $\Delta TV_{f,PT}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,1}^{\downarrow}(\bar{t})$.

7. *If the interaction is* **2-1/PT**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$, $\Delta TV_{f,PT}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,1}^{\downarrow}(\bar{t})$ *and* $\Delta TV_{f,PT}^{\uparrow}(\bar{t}) \le -\Delta TV_{f,2}^{\uparrow}(\bar{t})$.

8. *If the interaction is* **2-1/PT-1**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$, $\Delta TV_{f,1}^{\downarrow}(\bar{t}) < 0$ *and* $\Delta TV_{f,PT}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,1}^{\downarrow}(\bar{t})$.

9. *If the interaction is* **2-1/PT-2**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$, $\Delta TV_{f,2}^{\uparrow}(\bar{t}) < 0$, $\Delta TV_{f,PT}^{\uparrow}(\bar{t}) \le -\Delta TV_{f,2}^{\uparrow}(\bar{t})$ *and* $\Delta TV_{f,PT}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,1}^{\downarrow}(\bar{t})$.

10. *If the interaction is* **2-1/PT-1-2**, *then* $\Delta Tot.Var._{1,2}^{f}(\bar{t}) \le 0$, $\Delta TV_{f,1}^{\downarrow}(\bar{t}) < 0$, $\Delta TV_{f,2}^{\uparrow}(\bar{t}) < 0$ *and* $\Delta TV_{f,PT}^{\downarrow}(\bar{t}) \le -\Delta TV_{f,1}^{\downarrow}(\bar{t})$.

The proof can be found in the Appendix. Summarizing the results of Proposition 4.6, exchanges of flux variations among waves of the two Lax families and phase transitions may occur only according to the scheme of Figure 7.

PSfrag replacements
$1\uparrow$
$2\uparrow$
$PT\uparrow$
$1\downarrow$
$2\downarrow$
$PT\downarrow$

Figure 7: The possible exchanges of flux variation among waves for interactions involving a phase transition.

Corollary 4.1 *Let us consider the PT model (7)-(8) and an interaction involving a phase transition at time* $\bar{t}$. *Then* $\Delta TV_{f,1}^{\uparrow}(\bar{t}) \le 0$ *and* $\Delta TV_{f,2}^{\downarrow}(\bar{t}) \le 0$; *hence at time* $\bar{t}$ *no wave of the first family with increasing flux and no wave of the second family with decreasing flux can be generated if such waves are not present before the interaction. Moreover, if a wave of the first family with increasing flux or a wave of the second family with decreasing flux gives part of its flux variation to another type of wave, then it disappear.*

The proof is contained in the Appendix. The next proposition gives a bound on the total variation of the flux.

Proposition 4.7 *Assume that the phase transition system in (19) satisfies assumption **(H-2)** and the initial condition (ρ_r, q_r) satisfies the assumption **(H-1)** in $]0,+\infty[$ in the sense of Definitions 3.1 and 3.2. Then, for a.e. $t > 0$, we have*

$$Tot.Var._{1,2}^{f}(t) \leq Tot.Var._{1,2}^{f}(0) \cdot \Big(1 + K\, Tot.Var._{1,2}^{f}(0)\Big) \cdot \exp\Big(K\left(1 + K\, Tot.Var._{1,2}^{f}(0)\right) Tot.Var._{1,2}^{f}(0)\Big), \quad (45)$$

where $K > 0$ is the constant, introduced in Proposition 4.5.

Proof. We first briefly recall some important results.

1. By Remark 9, an increment of the total variation happens only when a wave of the first family with increasing flux interacts at a point $x > 0$ with a wave of the second family with decreasing flux. In this case the variation of $\mathrm{Tot.Var.}_{1,2}^{f}$ is bounded by K times the flux variation of the interacting wave of the first family.

2. A wave of the first family with increasing flux can increment its flux variation only if it interacts with a wave of the second family with decreasing flux and, viceversa, a wave of the second family with decreasing flux can increment its flux variation only if it interacts with a wave of the first family with increasing flux.

   Indeed in the case of an interaction at a point $x > 0$ involving a phase transition, by Proposition 4.6, no wave can transfer part of its flux variation to a wave of the first family with increasing flux or to a wave of the second family with decreasing flux; see also Figure 7. Moreover, in the case of an interaction at a point $x > 0$ involving a phase transition, by Corollary 4.1, if a wave of the first family with increasing flux or a wave of the second family with decreasing flux transfers part of its flux variation to another wave, then it disappears. Hence the statement of this point easily follows.

3. In the case of an interaction at a point $x > 0$ without phase transitions, a wave of the first family with increasing flux can transfer part of its flux variation only to a wave of the second family with increasing flux and a wave of the second family with decreasing flux can transfer part of its flux variation only to a wave of the first family with decreasing flux.

4. Waves of the first family with increasing flux can be generated only at time $t = 0$. Moreover, by the previous observations, one easily also deduces that $\mathrm{TV}_{f,1}^{\uparrow}(t) \leq (K+1)\,\mathrm{TV}_{f,1}^{\uparrow}(0+)$ for a.e. $t > 0$.

5. Waves of the second family with decreasing flux can be generated only at time $t = 0$ or at the interface $x = 0$. In the latter case, it can be due to a wave in the LWR model interacting with $x = 0$ or to an interaction with $x = 0$ of a wave of the first family with decreasing flux or of a phase transition wave with decreasing flux; see Lemma 4.5.

6. Waves of the first family with decreasing flux cannot be generated at $x = 0$. They can receive flux variation from waves of the second family with decreasing flux or from phase transition waves with decreasing flux; see Proposition 4.6 and Figure 7.

7. Phase transition waves with decreasing flux cannot be generated at $x = 0$. Moreover they can receive flux variation from waves of the second family with decreasing flux or from waves of the first family with decreasing flux; see Proposition 4.6 and Figure 7.

The maximum possible increment of the functional $\mathrm{Tot.Var.}_{1,2}^{f}$ can happen if the following occurs.

a) Every wave of the first family with increasing flux interacts with each wave of the second family with decreasing flux; see Figure 6. Moreover, all the waves in the LWR model interact also with $x = 0$. These interactions produce waves of the second family with decreasing flux, whose flux variation is lower than or equal to that of the interacting wave; see Lemma 4.5. These produced waves interact with all the remaining waves of the first family with increasing flux.

   This produces an increase of at most a factor $(1 + K\,\mathrm{Tot.Var.}_{1,2}^{f}(0))$.

b) Each wave of the second family with decreasing flux transfer its flux variation to a wave of the first family with decreasing flux or to a phase transition wave with decreasing flux. These two types of waves can eventually exchange part of their flux variation between them. Then they interact with the interface $x = 0$. These interactions produce waves of the second family with decreasing flux, whose flux variation is lower than or equal to that of the interacting wave; see Lemma 4.5. These produced waves interact with all the remaining waves of the first family with increasing flux; see Figure 8.

PSfrag replacements

x
t
$x = 0$
$1 \uparrow$
$2 \downarrow$
$1 \downarrow$ or $PT \downarrow$

PSfrag replacements

$2 \downarrow$
$1 \uparrow$
$1 \downarrow$
$PT \downarrow$
$x = 0$

Figure 8: Situation of the maximum possible increment of the total flux variation. Left: interaction with the interface producing a wave of the second family with decreasing flux, which contributes to an increment of the total variation. Right: scheme of possible exchange of flux variation.

Note that b) can repeat an arbitrary number of times, however when a wave of the first family with increasing flux interacts with a wave of the second family with decreasing flux, the produced waves can not interact together anymore. Now, if C indicates the flux variation of first family waves with increasing flux, then an interaction of the whole variation at once would produce an increase of $1 + KC$, while two successive interactions with half of the variation involved would create an increase of $(1+K\frac{C}{2})\cdot(1+K\frac{C}{2}) > 1+KC$. Repeating the reasoning we get the highest variation is achieved by using infinitesimal variations at each interaction, thus getting a maximal increase of a factor $\exp(K\,C)$. Taking into account that a) may occur first and then b) with the maximal increase just computed, we get the conclusion. □

4.2.3 Estimate for the functional $W_{1,2}$

In this part we want to give a uniform estimate for the functional $W_{1,2}$, defined in equation (37). In the sequel, we use the symbols 1, 2 and PT to indicate respectively a wave of the first, second family and a phase transition; see also Appendix A. First, notice that, by Lemma 4.2, if an interaction happens in the LWR system, then the functional $W_{1,2}$ does not increase.

Lemma 4.6 *Assume that the initial condition (ρ_r, q_r) satisfies the assumption* ***(H-1)*** *in $]0, +\infty[$ in the sense of Definition 3.1. Then there exist $0 < C_1 < C_2$ such that*

$$C_1\,Tot.Var._{1,2}^{f}(t) \le W_{1,2}(t) \le C_2\,Tot.Var._{1,2}^{f}(t) + N(t)V\,[\sigma_- + 1] \tag{46}$$

for a.e. $t > 0$, where $N(t)$ denotes the cardinality of $\mathcal{PT}_r(t)$.

Proof. Since hypothesis **(H-1)** holds, by Proposition 3.1, the functionals $\mathrm{Tot.Var.}_{1,2}^{f}$ and $W_{1,2}$ are equivalent if phase transition waves are not present. The equivalence does not hold, since it is possible to have phase transition waves with 0 contribution for $\mathrm{Tot.Var.}_{1,2}^{f}$, but with large contribution for $W_{1,2}$. The biggest possible contribution for $W_{1,2}$ in the case of a phase transition wave is $\omega\left(\sigma_-, q^-\sigma_-/R\right) - \omega(0,-1) + V = V(\sigma_- + 1)$; hence the conclusion follows. □

The next result proves that the functional $W_{1,2}$ does not increment for wave interaction inside the phase transition system; hence it can only increases for interaction of waves with the interface $x = 0$.

Lemma 4.7 *Assume that the waves $((\rho^l, q^l), (\rho^m, q^m))$ and $((\rho^m, q^m), (\rho^r, q^r))$ interact at the point $(\bar t, \bar x)$ with $\bar t > 0$ and $\bar x > 0$. Then $W_{1,2}(\bar t+) \le W_{1,2}(\bar t-)$.*

Proof. If the interaction does not involve any phase transition wave, then the conclusion is obvious. So let us consider only interactions involving phase transition waves. We use the same notation of Proposition 4.6 If the interaction is of type **2-PT/PT** (see Figure 11), then

$$\Delta W_{1,2}(\bar t) = \omega\left(\rho^m, q^m\right) - \omega\left(\rho^l, q^l\right) - \left|\omega\left(\rho^l, q^l\right) - \omega\left(\rho^m, q^m\right)\right| \le 0.$$

If the interaction is of type **2-PT/PT-1** (see Figure 12), then

$$\Delta W_{1,2}(\bar t) = 2\left(\omega\left(\rho^m, q^m\right) - \omega\left(\rho^l, q^l\right)\right) < 0.$$

If the interaction is of type **2-PT/1-2** (see Figure 12), then

$$\Delta W_{1,2}(\bar{t}) = 2\left(\omega\left(\rho^m, q^m\right) - \omega\left(\sigma_-, q^- \sigma_-/R\right)\right) < 0.$$

If the interaction is of type **PT-1/2** (see Figure 13), then

$$\Delta W_{1,2}(\bar{t}) = 2\left(v_c\left(\rho^m, q^m\right) - V\right) < 0.$$

If the interaction is of type **PT-1/PT** (see Figure 14), then

$$\Delta W_{1,2}(\bar{t}) = v_c\left(\rho^m, q^m\right) - v_c\left(\rho^r, q^r\right) - \left|v_c\left(\rho^m, q^m\right) - v_c\left(\rho^r, q^r\right)\right| \le 0.$$

If the interaction is of type **PT-1/PT-1** (see Figure 13), then $\Delta W_{1,2}(\bar{t}) = 0$.
If the interaction is of type **2-1/PT** (see Figure 15), then $\Delta W_{1,2}(\bar{t}) = 0$.
If the interaction is of type **2-1/PT-1** (see Figure 15), then $\Delta W_{1,2}(\bar{t}) = 0$.
If the interaction is of type **2-1/PT-2** (see Figure 16), then $\Delta W_{1,2}(\bar{t}) = 0$.
If the interaction is of type **2-1/PT-1-2** (see Figure 16), then $\Delta W_{1,2}(\bar{t}) = 0$.
The proof is so finished. □

Proposition 4.8 *Assume that the phase transition system in (19) satisfies assumption **(H-2)** in the sense of Definition 3.2 and that the initial condition (ρ_r, q_r) satisfies the assumption **(H-1)**. Then, for a.e. $t > 0$, we have*

$$W_{1,2}(t) \le M, \tag{47}$$

where $M > 0$ is a constant.

Proof. By Lemma 4.6, since the functional $\text{Tot.Var.}^f_{1,2}$ is uniformly bounded by Proposition 4.7, in order to obtain a bound for $W_{1,2}$ it is sufficient to estimate the number of the phase transition waves.

Note that, if a phase transition wave $\left((\rho^l, q^l), (\rho^r, q^r)\right)$ is generated at the interface, then (ρ^l, q^l), the trace at $x = 0+$, belongs to Ω_f. No other phase transition wave can be generated at $x = 0$ until the right trace at $x = 0$ of the wave front tracking approximate solution belongs to the free phase Ω_f. Therefore a new phase transition wave can be generated at $x = 0$ only after the wave $\left((\rho^l, q^l), (\rho^r, q^r)\right)$ is absorbed by the interface (after its speed changed sign) or disappears after interacting with another wave.

Obviously other phase transition waves can be generated by an interaction of waves at a point $x > 0$. However, by Lemma 4.7, this interaction does not produce an increment of $W_{1,2}$.

Therefore, if $\bar{N}$ is the number of phase transition waves at time $t = 0+$, then $W_{1,2}(t) \le K C_2 \text{Tot.Var.}^f_{1,2}(0+) + \left(\bar{N} + 1\right) V\left(\sigma_- + 1\right)$ for every positive time t, where K and C_2 are the constants introduced in the Proposition 4.5 and in Lemma 4.6. □

4.2.4 Existence of a wave-front tracking solution

We now want to bound the number of waves and of interactions.

Definition 4.6 *A wave of $\bar{u}_\varepsilon$, generated at time $t = 0$, is said an original wave or a wave with generation order* 1*. If a wave with generation order $k \ge 1$ interacts with the interface $x = 0$, then the produced waves are said of generation $k + 1$. If a wave with generation order $k \ge 1$ interacts at a point $x \ne 0$ with a wave with generation order $k' \ge 1$, then the produced wave is said of generation* $\min\{k, k'\}$.

Definition 4.7 *A wave of the second family $\left((\rho^l, q^l), (\rho^r, q^r)\right)$ is said special if $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) \in \Omega_f \cap \Omega_c \setminus \{(\sigma_-, \sigma_- q^-/R)\}$.*

Lemma 4.8 *Special waves can not emerge by interactions of waves inside the phase transition system. They can be generated only at time $t = 0$ or at the interface $x = 0$.*

Proof. By Proposition 4.6, special waves can not be generated by an interaction involving a phase transition wave. Moreover if a wave of the first family and a wave of the second family, both completely contained in Ω_c, interact together, then the resulting waves are not special and so the conclusion follows. □

Lemma 4.9 *Assume that the waves* $\left((\rho^l, q^l), (\rho^m, q^m)\right)$ *and* $((\rho^m, q^m), (\rho^r, q^r))$ *interacts together at a point* $x > 0$ *and at time* $\bar{t}$*. Suppose that a wave of the first family with increasing flux emerges from this interaction. Then the interaction is of type* **2-1/1-2** *and* $((\rho^m, q^m), (\rho^r, q^r))$ *is an original wave with increasing flux.*

PROOF. If the interaction involves a phase transition wave, then the emerging wave of the first family should have decreasing flux; see Proposition 4.6. Consequently, the interaction is of type **2-1/1-2**. This type of interaction does not change the behavior of waves and so the conclusion follows. □

Lemma 4.10 *The only waves coming back to the interface* $x = 0$ *from the LWR system are big shock. A big shock can come back to* $x = 0$ *only either interacting on the left with original waves of LWR or interacting to the right with waves with increasing flux produced by* $x = 0$.

The proof follows by [12, Lemma A.1.1 and Lemma A.1.2]. An immediate consequence is the following result.

Corollary 4.2 *Assume that a big shock, generated at* $x = 0$*, comes back to* $x = 0$ *interacting only with waves from the right. Then the waves from the right, which has increasing flux, are generated by the interaction from the phase transition model of waves of the first family with increasing flux.*

The following proposition holds.

Proposition 4.9 *For every* $\nu \in \mathbb{N} \setminus \{0\}$*, the construction in Subsection 4.2.1 can be done for every positive time, producing a* $\frac{1}{\nu}$*-approximate wave-front tracking solution to (19)-(27).*

PROOF. For $\nu \in \mathbb{N} \setminus \{0\}$, call $u_\nu = (\rho_{l,\nu}, (\rho_{r,\nu}, q_{r,\nu}))$ the function built with the procedure of Subsection 4.2.1. It is sufficient to prove that the number of waves and interactions, generated by the construction, is finite. Define the functions $N_{l,\nu}(t)$ and $N_{r,\nu}(t)$, which count the number of discontinuities respectively of $\rho_{l,\nu}$ and of $(\rho_{r,\nu}, q_{r,\nu})$. $N_{l,\nu}(t)$ and $N_{r,\nu}(t)$ are locally constant in time and can vary at interaction times in the following way.

1. If at time $\bar{t} > 0$ two waves interact at $\bar{x} < 0$, then $\Delta N_{l,\nu}(\bar{t}) = -1$ and $\Delta N_{r,\nu}(\bar{t}) = 0$.
2. If at time $\bar{t} > 0$ two waves interact at $\bar{x} > 0$, then $\Delta N_{l,\nu}(\bar{t}) = 0$ and $\Delta N_{r,\nu}(\bar{t}) \leq 1$. More precisely, $\Delta N_{r,\nu}(\bar{t}) = 1$ if and only if the interaction is of type **2-1/PT-1-2**.
3. If at time $\bar{t} > 0$ a wave interacts with the interface from the LWR model, then $\Delta N_{l,\nu}(\bar{t}) \leq 0$ and $\Delta N_{r,\nu}(\bar{t}) \leq 2$. Indeed at most one big shock is reflected in the LWR model, while both a of phase transition wave and a wave of the second family can be generated in the PT model.
4. If at time $\bar{t} > 0$ a wave interacts with the interface from the PT model, then $\Delta N_{l,\nu}(\bar{t}) \leq \nu R V_{max}/4$ and $\Delta N_{r,\nu}(\bar{t}) \leq 1$. The latter may happen only if a wave of first family with flux increase interacts with the interface.

The claims 1. to 3. are immediate. To prove 4., first notice that a rarefaction wave can be split into rarefaction fans in the LWR model, while both a phase transition wave and a wave of the second family can be generated in the PT model. However, the latter can be produced only by an interaction with the interface of a wave of the first family with flux increase. Indeed an interacting phase transition, which must have flux decrease, or a wave of first family with flux decrease and can produce only a wave of second family and no phase transition.

Now notice that the increment of $\nu R V_{max}/4$ for $N_{l,\nu}$ can happen at most a finite number of times. In fact, a rarefaction fan appears in the LWR model only if the data are good and if there is a flux increase. These can be produced only by an interaction with the interface of a wave of the first family with increasing flux. By Lemma 4.9 the number of waves of the first family with increasing flux can be bounded by $N_{r,\nu}(0+)$. Hence the number of waves in the LWR model is bounded.

Moreover, by Corollary 4.2 and Lemma 4.9 also the number of waves on the LWR system, generated by $x = 0$ and coming back to the interface, is bounded by $N_{r,\nu}(0+)$. Therefore the number of waves generated at $x = 0$ on the PT system is less than or equal to $2N_{l,\nu}(0+) + 2N_{r,\nu}(0+)$.

The number of waves in the PT system can increase also for the interaction **2-1/PT-1-2**. In this situation, a special wave interacts with a wave of the first family, producing three waves. Note that the wave of the second family after the interaction is not special. By Lemma 4.8, the number of special waves is bounded by

$3N_{r,\nu}(0+) + 2N_{l,\nu}(0+)$, i.e. the number of original waves in PT system plus the number of waves generated at $x = 0$ in the PT system. Hence

$$\begin{aligned} N_{r,\nu}(t) &\leq N_{r,\nu}(0+) + 2\left(N_{l,\nu}(0+) + N_{r,\nu}(0+)\right) + \left(3N_{r,\nu}(0+) + 2N_{l,\nu}(0+)\right) \\ &= 4N_{l,\nu}(0+) + 6N_{r,\nu}(0+) \end{aligned}$$

for a.e. $t > 0$.

Moreover the increment of waves in the LWR model are only due to the interaction of a wave in the PT model with $x = 0$, which can be original or not. By the previous considerations we obtain that

$$\begin{aligned} N_{l,\nu}(t) &\leq N_{l,\nu}(0+) + \frac{\nu R V_{max}}{4} N_{r,\nu}(0+) + \left(3N_{r,\nu}(0+) + 2N_{l,\nu}(0+)\right) \\ &\quad + 2\left(N_{l,\nu}(0+) + N_{r,\nu}(0+)\right) \\ &= 5N_{l,\nu}(0+) + \left(5 + \frac{\nu R V_{max}}{4}\right) N_{r,\nu}(0+), \end{aligned}$$

for a.e. $t > 0$.

Now we want to prove that the number of interactions is finite. By the previous analysis, we note that the number of interactions at the interface is finite. Moreover the number of interactions in the LWR model is finite, since each interaction destroys one wave. Let us focus to the PT system. The previous analysis showed that the interaction **2-1/PT-1-2** can happen at most a finite number of times. Since **2-1/PT-1-2** is the only case of increment of the number of waves in PT system, then also the interactions **1-1/1**, **2-PT/PT**, **PT-1/2**, **PT-1/PT**, **2-1/PT** can happen at most a finite number of times, since we have a uniform bound on the number of waves.

So it remains to consider the following interactions: **2-1/1-2**, **2-PT/PT-1**, **2-PT/1-2**, **PT-1/PT-1**, **2-1/PT-1**, **2-1/PT-2**. Notice that, if an emerging wave is a wave of second family, then a wave of the second family is present also before the interaction. Since each wave of the second family can interact at most once with phase transition waves and waves of the first family, then we deduce that interactions involving waves of the second family can happen at most a finite number of times. Therefore only **PT-1/PT-1** can happen an infinite number of times. In the $(\rho, \rho v)$ plane, the segment connecting the left and right state of the resulting phase transition wave is tangent to the lower boundary of Ω_c. This means that it can not interact on the right with a wave of the first family producing the interaction **PT-1/PT-1**; hence also **PT-1/PT-1** can happen a finite number of times. The proof is so concluded. □

4.2.5 Existence of a solution

In this part we conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. Fix an ε-approximate wave-front tracking solution $\bar{u}_\varepsilon$ to (19)-(27), in the sense of Definition 4.5. By Proposition 4.8, we deduce that there exists a constant $M > 0$, depending on the total variation of the flux of the initial datum, such that

$$W_{1,2}(t) \leq M,$$

for a.e. $t > 0$. In particular, we deduce that $\text{Tot.Var.} f\left(\bar{\rho}_{l,\varepsilon}(t,\cdot)\right) \leq M$ for a.e. $t > 0$; as in [13], we obtain that there exists a function $\bar{\rho}_l$, which is a solution to (19)-(27) for $x < 0$. Moreover, since **(H-1)** holds, then $W_{1,2}(t) \leq M$ for a.e. $t > 0$ implies that $\text{Tot.Var.}\left((\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})(t,\cdot)\right)$ is uniformly bounded for a.e. $t > 0$. Hence, at least by a subsequence, there is a function $(\bar{\rho}_{r,\varepsilon}, \bar{q}_{r,\varepsilon})$, which is a solution to (19)-(27) for $x > 0$. This permits to conclude. □

5 The PT-LWR model

This section deals with the coupling between the PT model if $x < 0$ and the LWR one if $x > 0$. More precisely we consider the following system

$$\begin{cases} \begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho, q)) = 0, & \text{if } (\rho, q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho, q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho, q)) = 0. \end{cases} & \text{if } (\rho, q) \in \Omega_c, \end{cases} & \text{if } x < 0,\ t > 0. \\ \partial_t \rho + \partial_x(\rho \tilde{v}(\rho)) = 0, & \text{if } x > 0,\ t > 0, \end{cases} \tag{48}$$

Definition 5.1 *The functions*

$$(\hat\rho_l, \hat q_l) \in C([0,+\infty[; L^1_{loc}(]-\infty,0]; \Omega_f \cup \Omega_c)) \qquad \text{and} \qquad \hat\rho_r \in C([0,+\infty[; L^1_{loc}([0,+\infty[; [0,R]))$$

are a weak solution to (48) if

1. *the function $(\hat\rho_l, \hat q_l)$ is a weak solution to*
$$\begin{cases} \partial_t \rho + \partial_x(\rho v_f(\rho,q)) = 0, & \text{if } (\rho,q) \in \Omega_f, \\ \begin{cases} \partial_t \rho + \partial_x(\rho v_c(\rho,q)) = 0, \\ \partial_t q + \partial_x(q v_c(\rho,q)) = 0. \end{cases} & \text{if } (\rho,q) \in \Omega_c, \end{cases}$$
for $(t,x) \in (0,+\infty) \times (-\infty, 0)$;

2. *the function $\hat\rho_r$ is a weak solution to*
$$\partial_t \rho + \partial_x(\rho \tilde v(\rho)) = 0$$
for $(t,x) \in (0,+\infty) \times (0,+\infty)$;

3. *for a.e. $t > 0$, the functions $x \mapsto (\hat\rho_l(t,x), \hat q_l(t,x))$ and $x \mapsto \hat\rho_r(t,x)$ both have versions with bounded total variation;*

4. *for a.e. $t > 0$, it holds*
$$\varphi\left(\hat\rho_l(t,0-), \hat q_l(t,0-)\right) = f(\hat\rho_r(t,0+)),$$
where $(\hat\rho_l, \hat q_l)$ and $\hat\rho_r$ stand for the versions with bounded total variation.

For functions
$$\begin{cases} (\hat\rho_l, \hat q_l) \in C([0,+\infty[; L^1_{loc}(]-\infty,0])) \\ \hat\rho_r \in C([0,+\infty[; L^1_{loc}([0,+\infty[)) \end{cases}$$
such that for a.e. $t > 0$ the maps $x \mapsto (\hat\rho_l(t,x), \hat q_l(t,x))$ and $x \mapsto \hat\rho_r(t,x)$ both have versions with bounded total variation, we define the functional

$$\text{Tot.Var.}^f_{1,2}(t) = \text{Tot.Var.}\varphi(\hat\rho_r(t,\cdot), \hat q_r(t,\cdot)) + \text{Tot.Var.}f(\hat\rho_l(t,\cdot)) \tag{49}$$

which is clearly defined for a.e. $t > 0$.

5.1 The Riemann Problem for PT-LWR coupled model

In this section we describe the Riemann problem for the PT-LWR model. More precisely we consider the system (48) with the piecewise initial conditions

$$\begin{cases} (\rho(0,x), q(0,x)) = (\rho_l, q_l), & \text{if } x < 0, \\ \rho(0,x) = \rho_r, & \text{if } x > 0, \end{cases} \tag{50}$$

where $\rho_r \in [0,R]$ and $(\rho_l, q_l) \in \Omega_f \cup \Omega_c$. Giving a solution to (48)-(50) is equivalent to prescribe a left and right trace at $x = 0$, denoted respectively by (ρ_l^b, q_l^b) and ρ_r^b. So we need to find $(\rho_l^b, q_l^b) \in \mathcal{T}_l^2(\rho_l, q_l)$ and $\rho_r^b \in \mathcal{T}_r^1(\rho_r)$ satisfying certain compatibility conditions.
First of all, we require the conservation of the number of vehicles passing through the boundary $x = 0$. The number of cars exiting from the region $x < 0$ per unit time is given by

$$\varphi(\rho_l^b, q_l^b) = \begin{cases} V\rho_l^b, & \text{if } (\rho_l^b, q_l^b) \in \Omega_f, \\ \frac{V\sigma}{R-\sigma}\rho_l^b\left(\frac{R}{\rho_l^b} - 1\right)\left(1 + q_l^b\right) & \text{if } (\rho_l^b, q_l^b) \in \Omega_c. \end{cases} \tag{51}$$

The number of vehicles entering in the region $x > 0$ per unit time is given by

$$\rho_r^b \tilde v(\rho_r^b) = \rho_r^b V_{max}\left(1 - \frac{\rho_r^b}{R}\right). \tag{52}$$

Thus the condition becomes

$$\rho_r^b V_{max}\left(1 - \frac{\rho_r^b}{R}\right) = \begin{cases} V\rho_l^b, & \text{if } (\rho_l^b, q_l^b) \in \Omega_f \\ \frac{V\sigma}{R-\sigma}\rho_l^b\left(\frac{R}{\rho_l^b} - 1\right)\left(1 + q_l^b\right), & \text{if } (\rho_l^b, q_l^b) \in \Omega_c. \end{cases} \tag{53}$$

Let us now define the concept of Riemann solver for (48)-(50).

Definition 5.2 *A Riemann solver for the Riemann problem (48)-(50) is a function*

$$\begin{array}{rcl}\mathcal{RS}_{2,1} : (\Omega_f \cup \Omega_c) \times [0,R] & \longrightarrow & (\Omega_f \cup \Omega_c) \times [0,R] \\ ((\rho_l, q_l), \rho_r) & \longmapsto & \left((\rho_l^b, q_l^b), \rho_r^b\right)\end{array}$$

such that:

1. $\rho_r^b \in \mathcal{T}_r^1(\rho_r)$ *and* $(\rho_l^b, q_l^b) \in \mathcal{T}_l^2(\rho_l, q_l)$*;*
2. $\rho_r^b V_{max}\left(1 - \frac{\rho_r^b}{R}\right) = \varphi(\rho_l^b, q_l^b)$*;*
3. $\mathcal{RS}_{2,1}\left(\mathcal{RS}_{2,1}\left((\rho_l, q_l), \rho_r\right)\right) = \mathcal{RS}_{2,1}\left((\rho_l, q_l), \rho_r\right)$.

We now characterize two different Riemann solvers for (48)-(50). Consider first the maximum flow, which can pass through the interface $x = 0$,

$$\Gamma_{(\rho_l,q_l),\rho_r} = \min\left\{ \sup_{\rho \in \mathcal{T}_r^1(\rho_r)} \rho V_{max}\left(1 - \frac{\rho}{R}\right), \sup_{(\rho,q)\in\mathcal{T}_l^2(\rho_l,q_l)} \varphi(\rho, q) \right\}. \tag{54}$$

The following lemma holds.

Lemma 5.1 *Fix* $\gamma \in [0, \Gamma_{(\rho_l,q_l),\rho_r}]$*, where* $\Gamma_{(\rho_l,q_l),\rho_r}$ *is defined in (54). There exist a unique* $\rho_r^b \in \mathcal{T}_r^1(\rho_r)$ *and a unique* $(\rho_l^b, q_l^b) \in \mathcal{T}_l^2(\rho_l, q_l)$ *such that*

$$\rho_r^b V_{max}\left(1 - \frac{\rho_r^b}{R}\right) = \gamma$$

and

$$\varphi(\rho_l^b, q_l^b) = \gamma.$$

PROOF. The lemma is based on the fact that the function φ is bijective on $\mathcal{T}_l^2(\rho_l, q_l)$ and that the function $\rho \mapsto \rho V_{max}\left(1 - \frac{\rho}{R}\right)$ is bijective on $\mathcal{T}_r^1(\rho_r)$. □

For the PT-LWR model we introduce the following definitions.

Definition 5.3 *Given a Riemann solver* $\mathcal{RS}_{2,1}$*, we say that* $((\rho_l, q_l), \rho_r)$ *is an equilibrium for* $\mathcal{RS}_{2,1}$ *if*

$$\mathcal{RS}_{2,1}((\rho_l, q_l), \rho_r) = ((\rho_l, q_l), \rho_r).$$

Definition 5.4 *Consider an equilibrium* $((\rho_l, q_l), \rho_r)$ *for* $\mathcal{RS}_{2,1}$*. We say that* ρ_r *provides a constraint for* $((\rho_l, q_l), \rho_r)$ *if*

$$f(\rho_r) = \sup_{\rho\in\mathcal{T}_r^1(\rho_r)} f(\rho).$$

We say that (ρ_l, q_l) *provides a constraint for* $((\rho_l, q_l), \rho_r)$ *if*

$$\varphi(\rho_l, q_l) = \sup_{(\rho,q)\in\mathcal{T}_l^2(\rho_l,q_l)} \varphi(\rho, q).$$

5.1.1 The Riemann solver which maximizes the flux

Here we construct the Riemann solver $\mathcal{RS}_{2,1}^1$ for (48)-(50), which maximizes the flux passing through the interface. The construction is done in the following way.

1. Given $\rho_r \in [0, R]$ and $(\rho_l, q_l) \in \Omega_f \cup \Omega_c$, define the maximum flow $\Gamma_{(\rho_l,q_l),\rho_r}$ as in equation (54).
2. By Lemma 5.1, there exist a unique $\rho_r^b \in \mathcal{T}_r^1(\rho_r)$ and a unique $(\rho_l^b, q_l^b) \in \mathcal{T}_l^2(\rho_l, q_l)$ such that

$$\rho_r^b V_{max}\left(1 - \frac{\rho_r^b}{R}\right) = \Gamma_{(\rho_l,q_l),\rho_r} \quad \text{and} \quad \varphi(\rho_l^b, q_l^b) = \Gamma_{(\rho_l,q_l),\rho_r}.$$

3. Define $\mathcal{RS}_{2,1}^1((\rho_l, q_l), \rho_r) = \left((\rho_l^b, q_l^b), \rho_r^b\right)$.

The following proposition holds.

Proposition 5.1 *The function* $\mathcal{RS}_{2,1}^1$ *is a Riemann solver for the Riemann problem (48)-(50).*

The proof is similar to that of Proposition 4.1; hence we omit it.

5.1.2 The Riemann solver with a flux constraint

Fix a positive constant $\bar{k} > 0$. Here we construct a Riemann solver $\mathcal{RS}^2_{2,1}$ for (48)-(50), which imposes a constraint on the flux passing through the interface. The construction is done in the following way.

1. Given $\rho_r \in [0, R]$ and $(\rho_l, q_l) \in \Omega_f \cup \Omega_c$, define the maximum flow $\Gamma_{(\rho_l,q_l),\rho_r}$ as in equation (54).
2. By Lemma 5.1, there exist a unique $\rho^b_r \in \mathcal{T}^1_r(\rho_r)$ and a unique $(\rho^b_l, q^b_l) \in \mathcal{T}^2_l(\rho_l, q_l)$ such that
$$\rho^b_r V_{max}\left(1 - \frac{\rho^b_r}{R}\right) = \min\left\{\bar{k}, \Gamma_{(\rho_l,q_l),\rho_r}\right\}$$
and
$$\varphi(\rho^b_l, q^b_l) = \min\left\{\bar{k}, \Gamma_{(\rho_l,q_l),\rho_r}\right\}$$
3. Define $\mathcal{RS}^2_{2,1}((\rho_l, q_l), \rho_r) = \left((\rho^b_l, q^b_l), \rho^b_r\right)$.

The following proposition holds.

Proposition 5.2 *Given $\bar{k} > 0$, the function $\mathcal{RS}^2_{2,1}$ is a Riemann solver for the Riemann problem (48)-(50).*

The proof is similar to that of Proposition 4.2.

5.2 The Cauchy problem

This subsection deals with the Cauchy problem for the PT-LWR coupled model. Fix $(\rho_l, q_l) \in BV(]-\infty, 0]; \Omega_f \cup \Omega_c)$ and $\rho_r \in BV([0, +\infty[; [0, R])$ and and consider the Cauchy problem for (48) with the initial conditions

$$\begin{cases} (\rho(0, x), q(0, x)) = (\rho_l(x), q_l(x)), & \text{if } x < 0, \\ \rho(0, x) = \rho_r(x), & \text{if } x > 0. \end{cases} \tag{55}$$

The main result for the PT-LWR model is the following theorem.

Theorem 5.1 *Fix the initial conditions $(\rho_l, q_l) \in BV(]-\infty, 0]; \Omega_f \cup \Omega_c)$ and $\rho_r \in BV([0, +\infty[; [0, R])$. Assume that $v_c(\rho_l(x), q_l(x)) \geq \bar{v}$ for a.e. $x < 0$, where $0 < \bar{v} < V$. Then there exists $((\hat{\rho}_l, \hat{q}_l), \hat{\rho}_r)$, weak solution to (48) in the sense of Definition 5.1, such that*

1. *$(\hat{\rho}_l(0, x), \hat{q}_l(0, x)) = (\rho_l(x), q_l(x))$ for a.e. $x < 0$;*
2. *$\hat{\rho}_r(0, x) = \rho_r(x)$ for a.e. $x > 0$;*
3. *for a.e. $t > 0$*
$$\mathcal{RS}_{2,1}((\hat{\rho}_l(t, 0-), \hat{q}_l(t, 0-)), \hat{\rho}_r(t, 0+)) = ((\hat{\rho}_l(t, 0-), \hat{q}_l(t, 0-)), \hat{\rho}_r(t, 0+)).$$

5.2.1 Wave-front tracking

As in Subsection 4.2.1, we are able to construct piecewise constant approximations via the wave-front tracking technique.

Definition 5.5 *Given $\varepsilon > 0$ and the Riemann solver $\mathcal{RS}_{2,1}$, we say that the map $\bar{u}_\varepsilon = ((\bar{\rho}_{l,\varepsilon}, \bar{q}_{l,\varepsilon}), \bar{\rho}_{r,\varepsilon})$ is an ε-approximate wave-front tracking solution to (48)-(55) if the following conditions hold.*

1. *It holds that*
$$\begin{cases} (\bar{\rho}_{l,\varepsilon}, \bar{q}_{l,\varepsilon}) \in C([0, +\infty[; L^1_{loc}([-\infty, 0[; \Omega_f \cup \Omega_c)). \\ \bar{\rho}_{r,\varepsilon} \in C([0, +\infty[; L^1_{loc}(]0, +\infty]; [0, R])) \end{cases}$$
2. *$(\bar{\rho}_{l,\varepsilon}, \bar{q}_{l,\varepsilon})$ is an ε-approximate wave-front tracking solution to the PT model on $x < 0$. Moreover the jumps can be entropic shocks, rarefaction shocks, phase-transition waves or contact discontinuities. They are indexed by $\mathcal{J}_r(t) = \mathcal{S}_r(t) \cup \mathcal{R}_r(t) \cup \mathcal{PT}_r(t) \cup \mathcal{CD}_r(t)$.*
3. *$\bar{\rho}_{r,\varepsilon}(t, x)$ is an ε-approximate wave-front tracking solution to (1) on $x > 0$. Moreover the jumps can be entropic shocks or rarefaction shocks and are indexed by $\mathcal{J}_l(t) = \mathcal{S}_l(t) \cup \mathcal{R}_l(t)$.*

PSfrag replacements

t

x

(ρ_l^-, q_l^-)

(ρ_l, q_l)

(ρ_l^+, q_l^+)

ρ_r^-

ρ_r^+

Figure 9: An interaction from the left for the PT-LWR model.

4. It holds that

$$\begin{cases} \|(\bar{\rho}_{l,\varepsilon}(0,\cdot), \bar{q}_{l,\varepsilon}(0,\cdot)) - (\rho_l(\cdot), q_l(\cdot))\|_{L^1(-\infty,0)} < \varepsilon \\ \mathit{Tot.Var.}\ (\bar{\rho}_{l,\varepsilon}(0,\cdot), \bar{q}_{l,\varepsilon}(0,\cdot)) \le \mathit{Tot.Var.}\ (\rho_l(\cdot), q_l(\cdot)) \\ \|\bar{\rho}_{r,\varepsilon}(0,\cdot) - \rho_r(\cdot)\|_{L^1(0,+\infty)} < \varepsilon \\ \mathit{Tot.Var.}\ \bar{\rho}_{r,\varepsilon}(0,\cdot) \le \mathit{Tot.Var.}\ \rho_r(\cdot). \end{cases}$$

5. For a.e. $t > 0$

$$\mathcal{RS}_{2,1}\left(\left(\bar{\rho}_{l,\varepsilon}, \bar{q}_{l,\varepsilon}\right)(t,0-), \bar{\rho}_{r,\varepsilon}(t,0+)\right) = \left(\left(\bar{\rho}_{l,\varepsilon}, \bar{q}_{l,\varepsilon}\right)(t,0-), \bar{\rho}_{r,\varepsilon}(t,0+)\right).$$

The construction of ε-approximate wave-front tracking solutions to (48)-(55) can be done in a similar way as in Subsection 4.2.1. Moreover we also assume that Remarks 4, 5 and 6 hold in this situation. Finally, we introduce in a similar way as in (34)-(37), the functional

$$W_{2,1}(t) = \mathrm{Tot.Var.} f\left(\bar{\rho}_{r,\varepsilon}(t,\cdot)\right) + W_1(t) + W_2(t) + W_{PT}(t). \tag{56}$$

5.2.2 Interaction estimates for the PT-LWR model

We consider here interaction estimates for waves hitting the interface $x = 0$ for the PT-LWR model. Let $\mathcal{RS}_{2,1}$ be the Riemann solver $\mathcal{RS}_{2,1}^1$, defined in Subsection 5.1.1, or the Riemann solver $\mathcal{RS}_{2,1}^2$, defined in Subsection 5.1.2. We have the following result in the case a wave interact at $x = 0$ from the left.

Proposition 5.3 *Let* $\left(\rho_l^-, q_l^-\right) \in [0,R]$, $\rho_r^- \in \Omega_f \cup \Omega_c$ *be such that*

$$\mathcal{RS}_{2,1}\left(\left(\rho_l^-, q_l^-\right), \rho_r^-\right) = \left(\left(\rho_l^-, q_l^-\right), \rho_r^-\right). \tag{57}$$

Assume that a wave $\left((\rho_l, q_l), \left(\rho_l^-, q_l^-\right)\right)$ *interacts with the interface* $x = 0$ *at a time* $\bar{t} > 0$. *Then*

$$\mathit{Tot.Var.}^f_{2,1}(\bar{t}-) = \mathit{Tot.Var.}^f_{2,1}(\bar{t}+). \tag{58}$$

Proof. Define $\left(\rho_l^+, q_l^+\right) \in \Omega_f \cup \Omega_c$ and $\rho_r^+ \in [0,R]$ by the relation

$$\left(\left(\rho_l^+, q_l^+\right), \rho_r^+\right) = \mathcal{RS}_{2,1}\left((\rho_l, q_l), \rho_r^-\right);$$

see Figure 9. Note that $\varphi\left(\rho_l^-, q_l^-\right) = f(\rho_r^-)$ and $\varphi\left(\rho_l^+, q_l^+\right) = f(\rho_r^+)$. We clearly have that

$$\begin{aligned} \mathrm{Tot.Var.}^f_{1,2}(\bar{t}+) - \mathrm{Tot.Var.}^f_{1,2}(\bar{t}-) &= \left|\varphi(\rho_l, q_l) - \varphi\left(\rho_l^+, q_l^+\right)\right| \\ &\quad - \left|\varphi(\rho_l, q_l) - \varphi\left(\rho_l^-, q_l^-\right)\right| + \left|f(\rho_r^-) - f(\rho_r^+)\right| \\ &= \left|\varphi(\rho_l, q_l) - \varphi\left(\rho_l^+, q_l^+\right)\right| - \left|\varphi(\rho_l, q_l) - \varphi\left(\rho_l^-, q_l^-\right)\right| \\ &\quad + \left|\varphi\left(\rho_l^-, q_l^-\right) - \varphi\left(\rho_l^+, q_l^+\right)\right|. \end{aligned} \tag{59}$$

There are the following possibilities.

PSfrag replacements

t

x

(ρ_l^-, q_l^-)

(ρ_l^+, q_l^+)

ρ_r

ρ_r^-

ρ_r^+

Figure 10: An interaction from the right for the PT-LWR model.

1. $\left(\rho_l^-, q_l^-\right)$ provides a constraint for the equilibrium $\left(\left(\rho_l^-, q_l^-\right), \rho_r^-\right)$. In this case $\left(\rho_l^-, q_l^-\right) \in \Omega_f$ and, since the interacting wave has strictly positive speed, we also deduce that $(\rho_l, q_l) \in \Omega_f$ and $\varphi(\rho_l, q_l) \neq \varphi\left(\rho_l^-, q_l^-\right)$.
   If $\varphi(\rho_l, q_l) < \varphi\left(\rho_l^-, q_l^-\right)$, then we conclude that $\left(\rho_l^+, q_l^+\right) = (\rho_l, q_l)$ and so (59) is equal to 0.
   If $\varphi(\rho_l, q_l) > \varphi\left(\rho_l^-, q_l^-\right)$, then the set $\Gamma_{(\rho_l^-, q_l^-), \rho_r^-}$ is contained in $\Gamma_{(\rho_l, q_l), \rho_r^-}$ (also the equality is possible). Therefore we deduce that $\varphi\left(\rho_l^-, q_l^-\right) \leq \varphi\left(\rho_l^+, q_l^+\right) \leq \varphi(\rho_l, q_l)$ and so (59) is equal to 0.

2. $\left(\rho_l^-, q_l^-\right)$ is not a constraint for the equilibrium $\left(\left(\rho_l^-, q_l^-\right), \rho_r^-\right)$. In this case $\left(\rho_l^-, q_l^-\right) \in \Omega_c \setminus \Omega_f$. Moreover the wave $\left((\rho_l, q_l), \left(\rho_l^-, q_l^-\right)\right)$ is either a wave of the second family completely contained in Ω_c or a phase transition wave.
   If the wave $\left((\rho_l, q_l), \left(\rho_l^-, q_l^-\right)\right)$ is completely contained in Ω_c and
   $$\sup_{(\rho,q) \in \mathcal{T}_l^2\left(\left(\rho_l^-, q_l^-\right)\right)} \varphi(\rho, q) \geq \varphi\left(\rho_l^-, q_l^-\right),$$
   then the solutions for the flux before and after the interaction coincide since $\left(\rho_l^-, q_l^-\right)$ is not a constraint for the equilibrium $\left(\left(\rho_l^-, q_l^-\right), \rho_r^-\right)$, and so $\rho_r^+ = \rho_r^-$ and $\varphi\left(\rho_l^+, q_l^+\right) = \varphi\left(\rho_l^-, q_l^-\right)$. Hence (59) is equal to 0.
   If the wave $\left((\rho_l, q_l), \left(\rho_l^-, q_l^-\right)\right)$ is completely contained in Ω_c and
   $$\sup_{(\rho,q) \in \mathcal{T}_l^2\left(\left(\rho_l^-, q_l^-\right)\right)} \varphi(\rho, q) < \varphi\left(\rho_l^-, q_l^-\right),$$
   then $\left(\rho_l^+, q_l^+\right) \in \Omega_f \cap \Omega_c$ and $\varphi(\rho_l, q_l) < \varphi\left(\rho_l^+, q_l^+\right) < \varphi\left(\rho_l^-, q_l^-\right)$; thus (59) is equal to 0.
   If $\left((\rho_l, q_l), \left(\rho_l^-, q_l^-\right)\right)$ is a phase transition wave, then $(\rho_l, q_l) \in \Omega_f$ and $\varphi(\rho_l, q_l) < \varphi\left(\rho_l^-, q_l^-\right)$. Therefore the set $\Gamma_{(\rho_l, q_l), \rho_r^-}$ is strictly contained in $\Gamma_{(\rho_l^-, q_l^-), \rho_r^-}$ and so $\left(\rho_l^+, q_l^+\right) = (\rho_l, q_l)$; hence (59) is equal to 0.

The proof is completed. □

In the case of an interaction from the right we have the following proposition.

Proposition 5.4 *Let $\left(\rho_l^-, q_l^-\right) \in [0, R]$, $\rho_r^- \in \Omega_f \cup \Omega_c$ be such that*

$$\mathcal{RS}_{2,1}\left(\left(\rho_l^-, q_l^-\right), \rho_r^-\right) = \left(\left(\rho_l^-, q_l^-\right), \rho_r^-\right). \tag{60}$$

Assume that a wave (ρ_r^-, ρ_r) interacts with the interface $x = 0$ at a time $\bar{t} > 0$. Then

$$Tot.Var._{2,1}^{f}(\bar{t}-) = Tot.Var._{2,1}^{f}(\bar{t}+). \tag{61}$$

Proof. Define $\left(\rho_l^+, q_l^+\right) \in \Omega_f \cup \Omega_c$ and $\rho_r^+ \in [0, R]$ by the relation

$$\left(\left(\rho_l^+, q_l^+\right), \rho_r^+\right) = \mathcal{RS}_{2,1}\left(\left(\rho_l^-, q_l^-\right), \rho_r\right);$$

see Figure 10. Note that $\varphi\left(\rho_l^-, q_l^-\right) = f(\rho_r^-)$ and $\varphi\left(\rho_l^+, q_l^+\right) = f(\rho_r^+)$. We clearly have that

$$\begin{aligned}\text{Tot.Var.}^{f}_{1,2}(\bar{t}+) - \text{Tot.Var.}^{f}_{1,2}(\bar{t}-) &= \left|\varphi\left(\rho_l^-, q_l^-\right) - \varphi\left(\rho_l^+, q_l^+\right)\right| \\ &\quad - \left|f(\rho_r) - f(\rho_r^-)\right| + \left|f(\rho_r) - f(\rho_r^+)\right| \\ &= \left|f(\rho_r^-) - f(\rho_r^+)\right| - \left|f(\rho_r) - f(\rho_r^-)\right| + \left|f(\rho_r) - f(\rho_r^+)\right|.\end{aligned}$$

Moreover, since the wave (ρ_r^-, ρ_r) has strictly negative speed, then $\rho_r \geq \frac{R}{2}$ and, by (4) and by Proposition 2.1, we deduce that $f(\rho_r) \geq f(\rho_r^+)$. Therefore

$$\begin{aligned}\text{Tot.Var.}^{f}_{1,2}(\bar{t}+) - \text{Tot.Var.}^{f}_{1,2}(\bar{t}-) &= f(\rho_r) - f(\rho_r^+) - \left|f(\rho_r) - f(\rho_r^-)\right| \\ &\quad + \left|f(\rho_r^-) - f(\rho_r^+)\right|.\end{aligned}$$

We have two different possibilities.

1. $f(\rho_r^+) \geq f(\rho_r^-)$. In this case, since $f(\rho_r) \geq f(\rho_r^+)$, we easily deduce that (40) holds.
2. $f(\rho_r^+) < f(\rho_r^-)$. In this case we claim that $\rho_r = \rho_r^+$. Indeed, since $\rho_r \geq \frac{R}{2}$, then, by (4), $\rho_r^+ = \rho_r$ or $\rho_r^+ < R - \rho_r$. Assume, by contradiction, that $\rho_r^+ < R - \rho_r$ and so the set $\Gamma_{(\rho_l^-, q_l^-), \rho_r}$, should be bigger than $\Gamma_{(\rho_l^-, q_l^-), \rho_r^-}$; this contradicts $f(\rho_r^+) < f(\rho_r^-)$. Therefore $\rho_r = \rho_r^+$ and so (40) holds.

The proof is so finished. □

With similar considerations of those of Section 4.2.2 (in particular changing the role of the waves of the first family with increasing flux with that of the second family with decreasing flux) and of Appendix A, we deduce the following proposition.

Proposition 5.5 *Assume that the phase transition system in (48)-(55) satisfies assumption **(H-2)** in the sense of Definition 3.2. Then, for a.e. $t > 0$, we have*

$$\mathit{Tot.Var.}^{f}_{2,1}(t) \leq \mathit{Tot.Var.}^{f}_{2,1}(0) \cdot \left(1 + K\,\mathit{Tot.Var.}^{f}_{2,1}(0)\right) \cdot \exp\left(K\left(1 + K\,\mathit{Tot.Var.}^{f}_{2,1}(0)\right) \mathit{Tot.Var.}^{f}_{2,1}(0)\right), \quad (62)$$

where $K > 0$ is the constant introduced in Proposition 4.5.

5.2.3 Estimate for the functional $W_{2,1}$

In this part we want to give a uniform estimate for the functional $W_{2,1}$, defined in equation (56).

Lemma 5.2 *Assume that the waves $((\rho^l, q^l), (\rho^m, q^m))$ and $((\rho^m, q^m), (\rho^r, q^r))$ interact at the point $(\bar{t}, \bar{x})$ with $\bar{t} > 0$ and $\bar{x} < 0$. Then $W_{2,1}(\bar{t}+) \leq W_{2,1}(\bar{t}-)$.*

Lemma 5.3 *Assume that the initial condition (ρ_l, q_l) satisfies the assumption **(H-1)** in $]-\infty, 0[$ in the sense of Definition 3.1. Then there exist $0 < C_1 < C_2$ such that*

$$C_1\,\mathit{Tot.Var.}^{f}_{2,1}(t) \leq W_{2,1}(t) \leq C_2\,\mathit{Tot.Var.}^{f}_{2,1}(t) + N(t) V\left[\sigma_- + 1\right] \quad (63)$$

for a.e. $t > 0$, where $N(t)$ denotes the cardinality of $\mathcal{PT}_l(t)$.

The proofs are completely identical of those of Lemma 4.7 and Lemma 4.6; hence we omit them.

Proposition 5.6 *Assume that the phase transition system in (19)-(27) satisfies assumption **(H-2)** in the sense of Definition 3.2 and that the initial condition (ρ_l, q_l) satisfies the assumption **(H-1)**. Then, for a.e. $t > 0$, we have*

$$W_{2,1}(t) \leq M, \quad (64)$$

where $M > 0$ is a constant.

Proof. By Lemma 5.2, an increment of $W_{1,2}$ can happen only when there is an interaction at the interface $x = 0$. By Lemma 5.3 the functional $W_{2,1}$ is not equivalent of $\text{Tot.Var.}^{f}_{2,1}$, since the presence of phase transition waves.

Note that, if a phase transition wave $\left((\rho^l, q^l), (\rho^r, q^r)\right)$ is generated at the interface, then (ρ^r, q^r), the trace at $x = 0-$, belongs to Ω_c. No other phase transition wave can be generated at $x = 0$ until the left trace belongs to the congested phase Ω_c. Therefore a new generation of a phase transition wave can happen only if $\left((\rho^l, q^l), (\rho^r, q^r)\right)$ is absorbed by the interface (after its speed changed sign) or it disappears after interacting with another wave. This permits to conclude, by Lemma 5.3. □

5.2.4 Existence of a wave-front tracking solution

We now want to bound the number of waves and of interactions. We start with two technical lemmas.

Lemma 5.4 *Let $\left(\rho^l, q^l\right) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) = \left(\sigma_-, \frac{q^- \sigma_-}{R}\right)$. Assume that the wave $\left((\rho^l, q^l), (\rho^r, q^r)\right)$ interacts with the interface $x = 0$ at time $\bar{t} > 0$. Then no wave emerges from the interface $x = 0$ at time $\bar{t}$ in the PT model, while a single shock wave exits from $x = 0$ at time $\bar{t}$ in the LWR model.*

Proof. Before time $\bar{t}$, the state (ρ^r, q^r) provides a constraint for the Riemann solver at $x = 0$ according to Definition 5.4. Therefore, since $\varphi(\rho^l, q^l) < \varphi(\rho^r, q^r)$, then, after $\bar{t}$, $\left(\rho^l, q^l\right)$ provides a constraint for the Riemann solver at $x = 0$; hence no wave exits from $x = 0$ in the PT model. Moreover in the LWR model, a wave with positive speed and increasing flux is produced; such wave can only be a shock wave. □

Lemma 5.5 *Assume that the phase transition wave $\left((\rho^l, q^l), (\rho^r, q^r)\right)$ interacts with the interface $x = 0$ at time $\bar{t} > 0$. Then no wave emerges from the interface $x = 0$ at time $\bar{t}$ in the PT model, while a single shock wave exits from $x = 0$ at time $\bar{t}$ in the LWR model.*

Proof. Since the interacting phase transition wave has positive speed, then $\varphi(\rho^l, q^l) < \varphi(\rho^r, q^r)$ and so $\left(\rho^l, q^l\right)$ provides a constraint for the Riemann solver at $x = 0$ after time $\bar{t}$; hence no wave exits from $x = 0$ in the PT model. Moreover in the LWR model, a wave with positive speed and increasing flux is produced; such wave can only be a shock wave. □

The following proposition holds.

Proposition 5.7 *For every $\nu \in \mathbb{N} \setminus \{0\}$, the construction in Subsection 5.2.1 can be done for every positive time, producing a $\frac{1}{\nu}$-approximate wave-front tracking solution to (48)-(55).*

Proof. For $\nu \in \mathbb{N} \setminus \{0\}$, call $u_\nu = ((\rho_{r,\nu}, q_{r,\nu}), \rho_{l,\nu})$ the function built with the procedure of Subsection 5.2.1. It is sufficient to prove that the number of waves and interactions, generated by the construction, is finite. Define the functions $N_{l,\nu}(t)$ and $N_{r,\nu}(t)$, which count the number of discontinuities respectively of $(\rho_{l,\nu}, q_{l,\nu})$ and of $\rho_{r,\nu}$. $N_{l,\nu}(t)$ and $N_{r,\nu}(t)$ are locally constant in time and can vary at interaction times in the following way.

1. If at time $\bar{t} > 0$ two waves interact at $\bar{x} > 0$, then $\Delta N_{l,\nu}(\bar{t}) = 0$ and $\Delta N_{r,\nu}(\bar{t}) = -1$.
2. If at time $\bar{t} > 0$ two waves interact at $\bar{x} < 0$, then $\Delta N_{l,\nu}(\bar{t}) \leq 1$ and $\Delta N_{r,\nu}(\bar{t}) = 0$. More precisely, $\Delta N_{l,\nu}(\bar{t}) = 1$ if and only if the interaction is of type **2-1/PT-1-2**.
3. If at time $\bar{t} > 0$ a wave interacts with the interface from the LWR model, then $\Delta N_{l,\nu}(\bar{t}) \leq 1 + \nu\sigma_+ V$ and $\Delta N_{r,\nu}(\bar{t}) \leq 0$. Indeed at most one big shock is reflected in the LWR model, while both a phase transition wave and a wave of the first family (possibly a rarefaction) can be generated in the PT model.
4. If at time $\bar{t} > 0$ a wave interacts with the interface from the PT model, then $\Delta N_{l,\nu}(\bar{t}) \leq \nu\sigma_+ V$ and $\Delta N_{r,\nu}(\bar{t}) \leq \nu R V_{max}/4$. Indeed, if the interacting wave is a phase transition, then no wave emerge in the PT model and a one big shock is generated in the LWR model; see Lemma 5.5. Instead, if the interacting wave is of the second family, then rarefaction waves can emerge both in the PT and in the LWR model.

First of all, note that the increment of the number of waves at point 2, due to the interaction of type **2-1/PT-1-2**, can happen at most a finite number of times. Indeed, this interaction is due to special waves and, after the interaction, the wave of the second family is not special. Repeating the proof of Lemma 4.8 and since waves of the second family have positive speed, one can easily deduce that special waves can be generated only at time $t = 0$.

Moreover, the situation at point 4, which produces an increment of the number of waves, can happen at most a finite number of times. In fact, the interacting wave is of the second family and new waves of the second family are generated at time $t = 0$ or by the interaction **PT-1/2**. The wave of the second family produced by **PT-1/2** has the left state in $\Omega_f \setminus \Omega_c$ and the right state is $\left(\sigma_-, \frac{q^- \sigma_-}{R}\right)$. By Lemma 5.4, if such wave interacts with the interface at time $\tilde{t}$, then $\Delta N_{l,\nu}(\tilde{t}) = -1$ and $\Delta N_{r,\nu} = 1$ and the number of waves remains constant. However, such wave of the second family can also interact with other waves in the PT model before interacting with the interface. Since $\left(\sigma_-, \frac{q^- \sigma_-}{R}\right) \in \Omega_f \cap \Omega_c$, it can only interacts from the right with a wave of the first family, generating at most a phase transition wave and waves of the first family (not second family). This permits to conclude that waves of the second family generated by the interaction **PT-1/2** are not responsible of the increment of waves of point 4.

By the previous analysis, $N_{l,\nu}(0+)$ provides an upper bound for the number of times such that both situations at points 2 and 4 (with an increment of the number of waves) can happen. Thus we have that

$$N_{r,\nu}(t) \le N_{r,\nu}(0+) + \frac{\nu R V_{max}}{4} N_{l,\nu}(0+)$$

for a.e. $t > 0$. By the consideration at point 1, $N_{r,\nu}(0+) + \frac{\nu R V_{max}}{4} N_{l,\nu}(0+)$ is an upper bound for the number of times the situation at point 3 can happen. Hence

$$\begin{aligned} N_{l,\nu}(t) \quad &\le \quad N_{l,\nu}(0+) + (1 + \nu\sigma_+ V)\left(N_{r,\nu}(0+) + \frac{\nu R V_{max}}{4} N_{l,\nu}(0+)\right) \\ &\qquad + N_{l,\nu}(0+)\nu\sigma_+ V + N_{l,\nu}(0+) \end{aligned}$$

for a.e. $t > 0$.

It remains to prove that the number of interactions is finite. By the previous analysis, we deduce that the number of interactions at $x = 0$ from the right is finite and also the number of interactions at $x = 0$ from the left, producing an increment of the number of waves, is finite. With the same arguments of the proof of Proposition 4.9 and by Lemmas 5.4 and 5.5, we deduce that the number of interactions inside the PT model and inside the LWR model is finite. Hence also the number of interactions at $x = 0$ from the left is finite. This permits to conclude. □

5.2.5 Existence of a solution

In this part we conclude the proof of Theorem 5.1.

Proof of Theorem 5.1. Fix an ε-approximate wave-front tracking solution $\bar{u}_\varepsilon$ to (48)-(55), in the sense of Definition 5.5. By Proposition 5.6, we deduce that there exists a constant $M > 0$, depending on the total variation of the flux of the initial datum, such that

$$W_{2,1}(t) \le M,$$

for a.e. $t > 0$. In particular, we deduce that Tot.Var.$f\left(\bar{\rho}_{r,\varepsilon}(t, \cdot)\right) \le M$ for a.e. $t > 0$; as in [13], we obtain that there exists a function $\bar{\rho}_r$, which is a solution to (48)-(55) for $x > 0$. Moreover, since **(H-1)** holds, then $W_{2,1}(t) \le M$ for a.e. $t > 0$ implies that Tot.Var.$\left((\bar{\rho}_{l,\varepsilon}, \bar{q}_{l,\varepsilon})(t, \cdot)\right)$ is uniformly bounded for a.e. $t > 0$. Hence, at least by a subsequence, there is a function $(\bar{\rho}_{l,\varepsilon}, \bar{q}_{l,\varepsilon})$, which is a solution to (48)-(55) for $x < 0$. This permits to conclude. □

Appendix

A Interactions involving phase-transition waves

In this appendix we collect some technical facts about interactions involving a phase transition wave. There are ten types of interactions distinguished based on the type of interacting waves and those appearing after the interaction. We use the symbols 1, 2 and PT to indicate respectively a wave of first, second family and a phase transition. We also indicate by $((\rho^l, q^l), (\rho^m, q^m))$ and $((\rho^m, q^m), (\rho^r, q^r))$ the interacting waves. All the possible interactions of waves involving phase transitions are the following ones.

2-PT/PT $(\rho^l, q^l), (\rho^m, q^m) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$. A single phase transition $((\rho^l, q^l), (\rho^r, q^r))$ is produced; see Figure 11.

2-PT/PT-1 $(\rho^l, q^l), (\rho^m, q^m) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$. There exists $(\tilde{\rho}, \tilde{q})$, with $(\tilde{\rho}, \tilde{q}) = \psi_2^-(\tilde{\rho}, \tilde{q})$, such that a phase transition $((\rho^l, q^l), (\tilde{\rho}, \tilde{q}))$ is produced followed by a wave $((\tilde{\rho}, \tilde{q}), (\rho^r, q^r))$ of the first family; see Figure 12.

2-PT/1-2 $(\rho^l, q^l) \in \Omega_f \cap \Omega_c$, $(\rho^m, q^m) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$. There exists $(\tilde{\rho}, \tilde{q}) \in \Omega_c \setminus \Omega_f$ such that a wave of the first family $((\rho^l, q^l), (\tilde{\rho}, \tilde{q}))$ is produced followed by a wave $((\tilde{\rho}, \tilde{q}), (\rho^r, q^r))$ of the second family; see Figure 12.

PSfrag replacements

(ρ^l, q^l)
(ρ^m, q^m)
(ρ^r, q^r)
ρ
0
φ

Figure 11: The interaction **2-PT/PT**.

PSfrag replacements

(ρ^l, q^l)
$(\tilde{\rho}, \tilde{q})$
(ρ^m, q^m)
(ρ^r, q^r)
ρ
0
φ

Figure 12: At left the interaction **2-PT/PT-1**. At right the interaction **2-PT/1-2**.

PSfrag replacements

(ρ^l, q^l)
$(\tilde{\rho}, \tilde{q})$
(ρ^m, q^m)
(ρ^r, q^r)
ρ
0
φ

Figure 13: At left the interaction **PT-1/2**. At right the interaction **PT-1/PT-1**.

PSfrag replacements

(ρ^l, q^l)

$(\tilde\rho, \tilde q)$

(ρ^m, q^m)

(ρ^r, q^r)

ρ

0

φ

Figure 14: The interaction **PT-1/PT**.

PSfrag replacements

(ρ^l, q^l)

$(\tilde\rho, \tilde q)$

(ρ^m, q^m)

(ρ^r, q^r)

ρ

0

φ

Figure 15: At left the interaction **2-1/PT**. At right the interaction **2-1/PT-1**.

PT-1/2 $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^m, q^m) = \psi_2^-(\rho^m, q^m)$ and $(\rho^r, q^r) = (\sigma_-, \frac{q^-\sigma_-}{R})$. A single wave $((\rho^l, q^l), (\rho^r, q^r))$ of the second family is produced; see Figure 13.

PT-1/PT $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^m, q^m) = \psi_2^-(\rho^m, q^m)$ and $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$, $v_c(\rho^r, q^r) < V$. A single phase transition $((\rho^l, q^l), (\rho^r, q^r))$ is produced; see Figure 14.

PT-1/PT-1 $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^m, q^m) = \psi_2^-(\rho^m, q^m)$ and $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$, $v_c(\rho^r, q^r) < V$. There exists $(\tilde\rho, \tilde q)$, with $(\tilde\rho, \tilde q) = \psi_2^-(\tilde\rho, \tilde q)$, such that a phase transition $((\rho^l, q^l), (\tilde\rho, \tilde q))$ is produced followed by a wave of the first family $((\tilde\rho, \tilde q), (\rho^r, q^r))$; see Figure 13.

2-1/PT $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^m, q^m) = (\sigma_-, \frac{q^-\sigma_-}{R})$ and $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$. A single phase transition wave $((\rho^l, q^l), (\rho^r, q^r))$ is produced; see Figure 15.

2-1/PT-1 $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^m, q^m) = (\sigma_-, \frac{q^-\sigma_-}{R})$ and $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$. There exists $(\tilde\rho, \tilde q)$, with $(\tilde\rho, \tilde q) = \psi_2^-(\tilde\rho, \tilde q)$, such that a phase transition $((\rho^l, q^l), (\tilde\rho, \tilde q))$ is produced followed by a wave $((\tilde\rho, \tilde q), (\rho^r, q^r))$ of the first family; see Figure 15.

2-1/PT-2 $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^m, q^m) \in \Omega_f \cap \Omega_c$ and $(\rho^r, q^r) \in \Omega_c \setminus \Omega_f$. There exists $(\tilde\rho, \tilde q)$, with $(\tilde\rho, \tilde q) = \psi_2^-(\tilde\rho, \tilde q)$, such that a phase transition $((\rho^l, q^l), (\tilde\rho, \tilde q))$ is produced followed by a wave $((\tilde\rho, \tilde q), (\rho^r, q^r))$ of the second family; see Figure 16.

2-1/PT-1-2 $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^m, q^m) \in \Omega_f \cap \Omega_c$ and $(\rho^r, q^r) \in \Omega_c \setminus \Omega_f$. There exists $(\tilde\rho, \tilde q)$, with $(\tilde\rho, \tilde q) = \psi_2^-(\tilde\rho, \tilde q)$ and $(\bar\rho, \bar q)$, with $(\bar\rho, \bar q) = \psi_2^-(\bar\rho, \bar q)$, such that a phase transition $((\rho^l, q^l), (\tilde\rho, \tilde q))$ is produced followed by a wave $((\tilde\rho, \tilde q), (\bar\rho, \bar q))$ of the first family and a wave $((\bar\rho, \bar q), (\rho^r, q^r))$ of the second family; see Figure 16.

Now we provide the technical proofs of two results of Section 4. Proof of Proposition 4.6. We denote by (ρ^l, q^l), (ρ^m, q^m) and (ρ^r, q^r) the states defining the interacting waves at $\bar t$.

Let us consider case 1; i.e. the interaction is of type **2-PT/PT**. We easily get that $(\rho^l, q^l), (\rho^m, q^m) \in \Omega_f \setminus \Omega_c$ and $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$; see Figure 11. By the triangular inequality we have that $\mathrm{Tot.Var.}_{1,2}^f(\bar t+) \leq \mathrm{Tot.Var.}_{1,2}^f(\bar t-)$ and so $\Delta\mathrm{Tot.Var.}_{1,2}^f(\bar t) \leq 0$. Moreover note that both $\Delta\mathrm{TV}_{f,2}^{\uparrow}(\bar t)$ and $\Delta\mathrm{TV}_{f,2}^{\downarrow}(\bar t)$ are less than or equal to 0, since the interaction does not produce any waves of the second family.

PSfrag replacements

(ρ^l, q^l)

$(\tilde\rho, \tilde q)$

$(\bar\rho, \bar q)$

(ρ^m, q^m)

(ρ^r, q^r)

ρ

0

φ

Figure 16: At left the interaction **2-1/PT-2**. At right the interaction **2-1/PT-1-2**.

If $\varphi(\rho^m, q^m) > \varphi(\rho^l, q^l) > \varphi(\rho^r, q^r)$, then $\Delta\mathrm{TV}^{\uparrow}_{f,PT}(\bar t) = 0$, $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = \varphi(\rho^l, q^l) - \varphi(\rho^m, q^m) < 0$ and so the conclusion follows.
If $\varphi(\rho^l, q^l) \geq \varphi(\rho^m, q^m) > \varphi(\rho^r, q^r)$, then $\Delta\mathrm{TV}^{\uparrow}_{f,PT}(\bar t) = 0$, $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = \varphi(\rho^l, q^l) - \varphi(\rho^m, q^m) = -\Delta\mathrm{TV}^{\downarrow}_{f,2}(\bar t)$ and so the conclusion follows.
If $\varphi(\rho^l, q^l) > \varphi(\rho^r, q^r) \geq \varphi(\rho^m, q^m)$, then $\Delta\mathrm{TV}^{\uparrow}_{f,PT}(\bar t) = \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r) \leq 0$, $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = \varphi(\rho^l, q^l) - \varphi(\rho^r, q^r) \leq \varphi(\rho^l, q^l) - \varphi(\rho^m, q^m) = -\Delta\mathrm{TV}^{\downarrow}_{f,2}(\bar t)$ and so the conclusion follows.
If $\varphi(\rho^m, q^m) \geq \varphi(\rho^r, q^r) \geq \varphi(\rho^l, q^l)$, then $\Delta\mathrm{TV}^{\uparrow}_{f,PT}(\bar t) = \varphi(\rho^r, q^r) - \varphi(\rho^l, q^l) \leq \varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) = -\Delta\mathrm{TV}^{\uparrow}_{f,2}(\bar t)$, $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = \varphi(\rho^r, q^r) - \varphi(\rho^m, q^m) \leq 0$ and so the conclusion follows.
If $\varphi(\rho^r, q^r) \geq \varphi(\rho^m, q^m) \geq \varphi(\rho^l, q^l)$, then $\Delta\mathrm{TV}^{\uparrow}_{f,PT}(\bar t) = \varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) = -\Delta\mathrm{TV}^{\uparrow}_{f,2}(\bar t)$, $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = 0$ and so the conclusion follows.
If $\varphi(\rho^r, q^r) \geq \varphi(\rho^l, q^l) \geq \varphi(\rho^m, q^m)$, then $\Delta\mathrm{TV}^{\uparrow}_{f,PT}(\bar t) = \varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) \leq 0$, $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = 0$ and so the conclusion follows.

Let us consider case 2; i.e. the interaction is of type **2-PT/PT-1**. We easily get that $(\rho^l, q^l), (\rho^m, q^m) \in \Omega_f \setminus \Omega_c$, $(\tilde\rho, \tilde q) = \psi_2^-(\tilde\rho, \tilde q)$, $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$ and $\varphi(\rho^l, q^l) > \varphi(\rho^m, q^m)$; see Figure 12. By the proof of Proposition 4.5 we have that $\mathrm{Tot.Var.}^f_{1,2}(\bar t+) \leq \mathrm{Tot.Var.}^f_{1,2}(\bar t-)$ and so $\Delta\mathrm{Tot.Var.}^f_{1,2}(\bar t) \leq 0$. Note that $\Delta\mathrm{TV}^{\downarrow}_{f,2}(\bar t) = \varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) < 0$ and $\Delta\mathrm{TV}^{\downarrow}_{f,1}(\bar t) = \varphi(\tilde\rho, \tilde q) - \varphi(\rho^r, q^r) > 0$.
If $\varphi(\rho^m, q^m) \geq \varphi(\rho^r, q^r)$, then $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = \varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q) + \varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)$ and so $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) + \Delta\mathrm{TV}^{\downarrow}_{f,1}(\bar t) = \varphi(\rho^l, q^l) - \varphi(\rho^m, q^m)$ and the conclusion follows.
If $\varphi(\rho^m, q^m) < \varphi(\rho^r, q^r)$, then $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = \varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q)$ and therefore $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) + \Delta\mathrm{TV}^{\downarrow}_{f,1}(\bar t) = \varphi(\rho^l, q^l) - \varphi(\rho^r, q^r) < \varphi(\rho^l, q^l) - \varphi(\rho^m, q^m) = -\Delta\mathrm{TV}^{\downarrow}_{f,2}(\bar t)$; the conclusion follows.

Let us consider case 3; i.e. the interaction is of type **2-PT/1-2**. We easily get that $(\rho^l, q^l) \in \Omega_f \cap \Omega_c$, $(\rho^m, q^m) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$ and $\varphi(\rho^l, q^l) > \varphi(\rho^m, q^m)$. Finally there exists $(\tilde\rho, \tilde q) \in \Omega_c$, which is the middle state of the resulting two waves; see Figure 12. By the proof of Proposition 4.5 we have that $\mathrm{Tot.Var.}^f_{1,2}(\bar t+) \leq \mathrm{Tot.Var.}^f_{1,2}(\bar t-)$ and so $\Delta\mathrm{Tot.Var.}^f_{1,2}(\bar t) \leq 0$. Note that $\Delta\mathrm{TV}^{\downarrow}_{f,2}(\bar t) = \varphi(\tilde\rho, \tilde q) - \varphi(\rho^r, q^r) + \varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) < 0$ by **(H-2)** and $\Delta\mathrm{TV}^{\downarrow}_{f,1}(\bar t) = \varphi(\rho^l, q^l) - \varphi(\tilde\rho, \tilde q)$; hence $\Delta\mathrm{TV}^{\downarrow}_{f,2}(\bar t) + \Delta\mathrm{TV}^{\downarrow}_{f,1}(\bar t) = \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)$. If $\varphi(\rho^m, q^m) - \varphi(\rho^r, q^r) \leq 0$, then the conclusion easily follows. If $\varphi(\rho^m, q^m) - \varphi(\rho^r, q^r) > 0$, then $\Delta\mathrm{TV}^{\downarrow}_{f,PT} = \varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)$ and the conclusion follows.

Let us consider case 4; i.e. the interaction is of type **PT-1/2**. We easily get that $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) = \left(\sigma_-, \frac{q^- \sigma_-}{R}\right)$, $(\rho^m, q^m) = \psi_2^-(\rho^m, q^m)$; see Figure 13. By the triangular inequality we have that $\mathrm{Tot.Var.}^f_{1,2}(\bar t+) \leq \mathrm{Tot.Var.}^f_{1,2}(\bar t-)$ and so $\Delta\mathrm{Tot.Var.}^f_{1,2}(\bar t) \leq 0$. Note that $\Delta\mathrm{TV}^{\uparrow}_{f,PT}(\bar t) \leq 0$, since no phase transition waves are produced. Moreover $\Delta\mathrm{TV}^{\uparrow}_{f,2}(\bar t) = \varphi(\rho^r, q^r) - \varphi(\rho^l, q^l)$, $\Delta\mathrm{TV}^{\uparrow}_{f,1}(\bar t) = \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r) < 0$ and consequently $\Delta\mathrm{TV}^{\uparrow}_{f,2}(\bar t) + \Delta\mathrm{TV}^{\uparrow}_{f,1}(\bar t) = \varphi(\rho^m, q^m) - \varphi(\rho^l, q^l)$. If $\varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) \leq 0$, then the conclusion easily follows. If $\varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) > 0$, then $\Delta\mathrm{TV}^{\uparrow}_{f,PT} = \varphi(\rho^l, q^l) - \varphi(\rho^m, q^m)$ and the conclusion follows.

Let us consider case 5; i.e. the interaction is of type **PT-1/PT**. We easily get that $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$, $(\rho^m, q^m) = \psi_2^-(\rho^m, q^m)$; see Figure 14. By the triangular inequality we have that $\mathrm{Tot.Var.}^f_{1,2}(\bar t+) \leq \mathrm{Tot.Var.}^f_{1,2}(\bar t-)$ and so $\Delta\mathrm{Tot.Var.}^f_{1,2}(\bar t) \leq 0$. Note that $\Delta\mathrm{TV}^{\uparrow}_{f,1}(\bar t) \leq 0$ and $\Delta\mathrm{TV}^{\downarrow}_{f,1}(\bar t) \leq 0$, since no waves of the first family are produced.
If $\varphi(\rho^l, q^l) \leq \varphi(\rho^m, q^m) < \varphi(\rho^r, q^r)$, then $\Delta\mathrm{TV}^{\downarrow}_{f,PT}(\bar t) = 0$, $\Delta\mathrm{TV}^{\uparrow}_{f,PT}(\bar t) = \varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)$ and

$\Delta \mathrm{TV}_{f,1}^{\uparrow}(\bar{t}) = \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)$; hence the conclusion follows.
If $\varphi(\rho^m, q^m) < \varphi(\rho^l, q^l) \leq \varphi(\rho^r, q^r)$, then $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = \varphi(\rho^r, q^r) - \varphi(\rho^l, q^l)$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) < 0$, $\Delta \mathrm{TV}_{f,1}^{\uparrow}(\bar{t}) = \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)$; so the conclusion follows.
If $\varphi(\rho^m, q^m) < \varphi(\rho^r, q^r) < \varphi(\rho^l, q^l)$, then $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = 0$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r) < 0$; so the conclusion follows.
If $\varphi(\rho^l, q^l) \leq \varphi(\rho^r, q^r) < \varphi(\rho^m, q^m)$, then $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = \varphi(\rho^r, q^r) - \varphi(\rho^m, q^m) < 0$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = 0$; so the conclusion follows.
If $\varphi(\rho^r, q^r) < \varphi(\rho^l, q^l) \leq \varphi(\rho^m, q^m)$, then $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\rho^m, q^m) \leq 0$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)$ and $\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t}) = \varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)$; so the conclusion follows.
If $\varphi(\rho^r, q^r) < \varphi(\rho^m, q^m) < \varphi(\rho^l, q^l)$, then $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = 0$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r)$ and $\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t}) = \varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)$; so the conclusion follows.

Let us consider case 6; i.e. the interaction is of type **PT-1/PT-1**. We easily get that $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$, $(\rho^m, q^m) = \psi_2^-(\rho^m, q^m)$. Finally there exists $(\tilde{\rho}, \tilde{q}) = \psi_2^-(\tilde{\rho}, \tilde{q})$, which is the middle state of the resulting two waves; see Figure 13. By the proof of Proposition 4.5, we have that $\mathrm{Tot.Var.}_{1,2}^{f}(\bar{t}+) \leq \mathrm{Tot.Var.}_{1,2}^{f}(\bar{t}-)$ and so $\Delta\mathrm{Tot.Var.}_{1,2}^{f}(\bar{t}) \leq 0$. We easily get that $\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t}) = \varphi(\tilde{\rho}, \tilde{q}) - \varphi(\rho^m, q^m) < 0$.
If $\varphi(\rho^l, q^l) \leq \varphi(\rho^m, q^m)$, then $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\tilde{\rho}, \tilde{q})$ and the conclusion follows.
If $\varphi(\rho^l, q^l) > \varphi(\rho^m, q^m)$, then $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^m, q^m) - \varphi(\tilde{\rho}, \tilde{q})$ and the conclusion follows.

Let us consider case 7; i.e. the interaction is of type **2-1/PT**. We get that $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$, $(\rho^m, q^m) = \left(\sigma_-, \frac{q^- \sigma_-}{R}\right)$; see Figure 15. By the triangular inequality we have that $\mathrm{Tot.Var.}_{1,2}^{f}(\bar{t}+) \leq \mathrm{Tot.Var.}_{1,2}^{f}(\bar{t}-)$ and so $\Delta\mathrm{Tot.Var.}_{1,2}^{f}(\bar{t}) \leq 0$. We deduce that $\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t}) = \varphi(\rho^r, q^r) - \varphi(\rho^m, q^m) < 0$, $\Delta \mathrm{TV}_{f,2}^{\uparrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\rho^m, q^m) < 0$. If $\varphi(\rho^l, q^l) \leq \varphi(\rho^r, q^r)$, then $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = \varphi(\rho^r, q^r) - \varphi(\rho^l, q^l)$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = 0$ and the conclusion follows.
If $\varphi(\rho^l, q^l) > \varphi(\rho^r, q^r)$, then $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = 0$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)$ and the conclusion follows.

Let us consider case 8; i.e. the interaction is of type **2-1/PT-1**. We get that $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) = \psi_2^-(\rho^r, q^r)$, $(\rho^m, q^m) = \left(\sigma_-, \frac{q^- \sigma_-}{R}\right)$. Finally there exists $(\tilde{\rho}, \tilde{q}) = \psi_2^-(\tilde{\rho}, \tilde{q})$, which is the middle state of the resulting two waves; see Figure 15. We clearly have that $\Delta\mathrm{Tot.Var.}_{1,2}^{f}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\rho^r, q^r) - (2\varphi(\rho^m, q^m) - \varphi(\rho^l, q^l) - \varphi(\rho^r, q^r)) < 0$. Moreover $\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t}) = \varphi(\tilde{\rho}, \tilde{q}) - \varphi(\rho^m, q^m) < 0$ and $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\tilde{\rho}, \tilde{q})$; hence the conclusion follows.

Let us consider case 9; i.e. the interaction is of type **2-1/PT-2**. We get that $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) \in \Omega_c$, $(\rho^m, q^m) \in \Omega_f \cap \Omega_c$. Finally there exists $(\tilde{\rho}, \tilde{q}) = \psi_2^-(\tilde{\rho}, \tilde{q})$, which is the middle state of the resulting two waves; see Figure 16. We clearly have $\Delta \mathrm{TV}_{f,2}^{\uparrow}(\bar{t}) = \varphi(\rho^r, q^r) - \varphi(\tilde{\rho}, \tilde{q}) - \varphi(\rho^m, q^m) + \varphi(\rho^l, q^l)$, which is strictly negative by **(H-2)**. Moreover $\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t}) < 0$.
If $\varphi(\rho^l, q^l) \leq \varphi(\tilde{\rho}, \tilde{q})$, then $\Delta\mathrm{Tot.Var.}(\bar{t}) = -\varphi(\rho^l, q^l) + \varphi(\rho^r, q^r) - 2\varphi(\rho^m, q^m) + \varphi(\rho^l, q^l) + \varphi(\rho^r, q^r) = 2(\varphi(\rho^r, q^r) - \varphi(\rho^m, q^m)) < 0$, $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = \varphi(\tilde{\rho}, \tilde{q}) - \varphi(\rho^l, q^l) \leq -\Delta \mathrm{TV}_{f,2}^{\uparrow}(\bar{t})$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = 0$ and the conclusion follows.
If $\varphi(\rho^l, q^l) > \varphi(\tilde{\rho}, \tilde{q})$, then $\Delta\mathrm{Tot.Var.}(\bar{t}) = 2\varphi(\rho^l, q^l) - 2\varphi(\tilde{\rho}, \tilde{q}) + 2\varphi(\rho^r, q^r) - 2\varphi(\rho^m, q^m) \leq 0$ by **(H-2)**, $\Delta \mathrm{TV}_{f,PT}^{\uparrow}(\bar{t}) = 0$, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\tilde{\rho}, \tilde{q}) \leq \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r) = -\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t})$ by **(H-2)**. Thus the conclusion follows.

Let us consider case 10; i.e. the interaction is of type **2-1/PT-1-2**. We get that $(\rho^l, q^l) \in \Omega_f \setminus \Omega_c$, $(\rho^r, q^r) \in \Omega_c$, $(\rho^m, q^m) \in \Omega_f \cap \Omega_c$. Finally there exist $(\tilde{\rho}, \tilde{q}) = \psi_2^-(\tilde{\rho}, \tilde{q})$, $(\bar{\rho}, \bar{q}) = \psi_2^-(\bar{\rho}, \bar{q})$, which are the middle state of the resulting two waves; see Figure 16. We clearly have that $\varphi(\rho^l, q^l) > \varphi(\tilde{\rho}, \tilde{q})$. Similarly to the previous case, we deduce that $\Delta\mathrm{Tot.Var.}(\bar{t}) = 2\varphi(\rho^l, q^l) - 2\varphi(\bar{\rho}, \bar{q}) + 2\varphi(\rho^r, q^r) - 2\varphi(\rho^m, q^m) \leq 0$ by **(H-2)**, $\Delta \mathrm{TV}_{f,PT}^{\downarrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\tilde{\rho}, \tilde{q}) \leq \varphi(\rho^m, q^m) - \varphi(\rho^r, q^r) - \varphi(\tilde{\rho}, \tilde{q}) + \varphi(\bar{\rho}, \bar{q}) = -\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t})$ by **(H-2)**. Finally $\Delta \mathrm{TV}_{f,1}^{\downarrow}(\bar{t}) = \varphi(\tilde{\rho}, \tilde{q}) - \varphi(\bar{\rho}, \bar{q}) - \varphi(\rho^m, q^m) + \varphi(\rho^r, q^r)$ and $\Delta \mathrm{TV}_{f,2}^{\uparrow}(\bar{t}) = \varphi(\rho^l, q^l) - \varphi(\bar{\rho}, \bar{q}) - \varphi(\rho^m, q^m) + \varphi(\rho^r, q^r)$, which are negative by **(H-2)**. Thus the conclusion follows.

The proof is so concluded. □

Proof of Corollary 4.1. The fact that $\Delta \mathrm{TV}_{f,1}^{\uparrow}(\bar{t}) \leq 0$ and $\Delta \mathrm{TV}_{f,2}^{\downarrow}(\bar{t}) \leq 0$ follows by checking the 10 possibilities in the proof of Proposition 4.6.
If a wave of the first family with increasing flux or a wave of the second family with decreasing flux gives part of its variation to another type of wave, then at time $\bar{t}$ only interactions of types 1, 2, 4, 5 can happen; see Figure 7. Therefore the conclusion easily follows. □

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