EXISTENCE OF SOLUTION TO SUPPLY CHAIN MODELS BASED ON PARTIAL DIFFERENTIAL EQUATION WITH DISCONTINUOUS FLUX FUNCTION

M. HERTY, CH. JÖRRES, AND B. PICCOLI

Abstract. We consider a recently [2] proposed model for supply chains with finite buffers. The continuous model is based on a conservation law with discontinuous flux function. A suitable reformulation of the model is introduced and studied analytically. The latter include waves with infinite negative speed. Using wave-front tracking existence of solutions are obtained for supply chains and production lines (consisting of sequences of supply chains.) Numerical results are also presented.

Keywords. supply chains, front tracking, discontinuous flux
AMS. 90B10, 65Mxx

1. Introduction

Nowadays, different approaches to supply chain modeling exists. On the one hand there are queuing models and discrete event models, on the other hand continuous models similar to gas dynamics equations have been studied. Recently, the modeling of supply chain problems using partial differential equations has being subject to many different publications, see for example [5, 1, 3, 6, 14]. A prototype model conserves the number of parts being processed. It furthers has storage capacity (buffers) for excess parts currently not processed. Hence, the arising equations are typically conservation laws possibly coupled to ordinary differential equations [12] or transport equations [6] for processing capacities. A main drawback of existing continuous models [1, 12] has been the assumption on unlimited buffers. In a recent publication a model for a supply chain with limited buffers [2] has been proposed. The time evolution of the part density $\rho(t,x) \in [0,1]$ within the supply chain, parametrized by $x \in [0,1]$, is given by

$$\partial_t \rho(t,x) + \partial_x \mathbf{F}(\rho(t,x)) = 0, \ \rho(0,x) = \rho_0(x), \tag{1}$$

and $\mathbf{F} : [0,1] \to [0,1]$ is *discontinuous* at maximal density. To be more precise, we assume $\mathbf{F}(0) = 0$ and that $\mathbf{F}$ is a smooth and strictly monotone increasing function on $[0,1)$ and is discontinuous at $x = 1$ with $\mathbf{F}(1) = 0$. The function $\mathbf{F}$ is called clearing function and $\rho = 1$ is the upper buffer limit of the supply chain, having zero throughput. The meaning is the following: If the buffer is full no more parts can be processed and the information of a buffer being full is transported immediately throughout the supply chain. For a particular choice of F the model (1) matches well with discrete event simulations of a supply chain with limited buffers, see [2, Section 6].

Flux functions with single discontinuities are usually approximated by Friedrichs mollifiers. By studying the limits one can obtain a solution for the discontinuous flux. This

Date: September 14, 2012.

has been studied for example in [9, 7, 8, 10] and in the context of a traffic flow model recently in [15]. Our approach differs from the previous results and the main achievement is an extension of results proposed in [2]. We present a reformulation of the model (1) keeping the same properties but allowing for an improved analytical treatment. More precisely we extend the state space including a variable $\mathcal{S}$ detecting the *status* or phase of the supply chain, which could be free or congested. The flux is then defined accordingly, with zero value for congested phase. The overall dynamics include waves with infinite negative speed, which however transport information only on the the variable $\mathcal{S}$. The latter may happen only at initial time for a single supply chain, but also at later time for production lines (consisting of a sequence of supply chains.) The dynamics on a whole production line is determined by defining soluton at nodes, which are compatible with the Riemann solution on a single supply chain in free phase.
We prove existence of solutions first for the case of a single supply chain and then for a production line. Approximate solutions are defined by a wave-front tracking approach and estimates are used on waves, interactions and the total variation of the solution.
The presence of waves with infinite negative speed generates new phenomena, which are illustrated by a numerical simulation. The latter is achieved by a numerical scheme, which combines the classical Godunov scheme with node dynamics and the change in the variable $\mathcal{S}$ due to infinte speed waves.

2. The model

We consider first the Cauchy problem for (1) on a real line and assume that the initial data $\rho_0(x) \in L^1(\mathbb{R})$ is non–negative and bounded by one, i.e., $\|\rho_0\|_\infty \leq 1$. To deal with such problem, we modify the dynamics (1) by introducing a new function F with arguments the density $\rho \in [0,1]$ and a second argument $\mathcal{S}$ attaining values in the finite set $\{\mathcal{F},\mathcal{C}\}$ only. The latter represent the *status* of the network, with $\mathcal{F}$ corresponding to free phase and $\mathcal{C}$ to congested phase. Define $F : [0,1] \times \{\mathcal{F},\mathcal{C}\} \to \mathbb{R}$ by

$$F(\rho,\mathcal{S}) = \begin{cases} \mathbf{F}(\rho) & \text{if} \quad 0 \leq \rho < 1, \quad \mathcal{S} = \mathcal{F} \\ \lim\limits_{\rho \to 1^-} \mathbf{F}(\rho) & \text{if} \quad \rho = 1, \quad \mathcal{S} = \mathcal{F} \\ \mathbf{F}(\rho) & \text{if} \quad 0 \leq \rho < 1, \quad \mathcal{S} = \mathcal{C} \\ 0 & \text{if} \quad \rho = 1, \quad \mathcal{S} = \mathcal{C} \end{cases} \tag{2}$$

On F we impose the following assumptions satisfied by the examples of [2]:

$$\rho \to \mathbf{F}(\rho) \text{ is smooth }, \partial_\rho \mathbf{F}(\rho) > 0 \;\forall \rho \text{ and } \mathbf{F}(0) = 0. \tag{3}$$

The evolution of $(\rho(t,x), \mathcal{S}(t,x))$ corresponding to (1) is given by a conservation law paired to a state constraint:

$$\partial_t \rho + \partial_x F(\rho,\mathcal{S}) = 0, \tag{4a}$$

$$\mathcal{S}(t,x) = \mathcal{C}(t,x) \implies \rho(t,x) = 1. \tag{4b}$$

The meaning of the state constraint is that congested phase can appear only at maximal density. The initial data are

$$\rho(0,x) = \rho_0(x), \mathcal{S}(0,x) = \mathcal{S}_0(x). \tag{5}$$

Next, we discuss some analytical properties of (4). Let J be an interval in $\mathbb{R}$. Define the total variation of a real function $V(x) : J \to \mathbb{R} \times \{\mathcal{F}, \mathcal{C}\}$ with $V(x) = (\rho, \mathcal{S})(x)$ as

$$TV(V) := \sup\left\{\sum_{i=1}^{N} \|\rho(x_i) - \rho(x_{i-1})\|\right\} + \sup\left\{\sum_{k=1}^{M} \begin{cases} 1 & \mathcal{S}(x_k) \neq \mathcal{S}(x_{k-1}) \\ 0 & \text{else} \end{cases}\right\},$$

where $N, M \geq 1$ and the points x_i and x_k belong to J and are ordered. Clearly, the first part corresponds to the total variation of the function ρ and we say V is of bounded variation if $TV(V) < \infty$. We denote by BV the set of all functions $V : J \to \mathbb{R} \times \{\mathcal{F}, \mathcal{C}\}$ with bounded total variation.

Denote by $U := (\rho, \mathcal{S})$. Consider the Riemann problem on $[0, \infty) \times \mathbb{R}$

$$(4) \text{ and } U_0(x) = \begin{cases} (\rho_l, \mathcal{S}_l) & x < 0 \\ (\rho_r, \mathcal{S}_r) & x > 0 \end{cases} \tag{6}$$

Allowing waves of infinite negative speeds (to connect with congested phase), we can solve Riemann problems as stated in next Lemma.

Lemma 2.1. *Let F be given by (2) and suppose (3) holds. Then, the Riemann problem (6) admits a solution to any initial data $U_l := (\rho_l, \mathcal{S}_l)$ and $U_r := (\rho_r, \mathcal{S}_r)$ with $\rho_l, \rho_r \in [0, 1]$ and $\mathcal{S}_l, \mathcal{S}_r \in \{\mathcal{F}, \mathcal{C}\}$.*

The solution $U(x,t)$ fulfills a maximum principle with respect to ρ and for all $t > 0$ we have $TV(U(\cdot,t)) \leq TV(U_0)$.

Note that density is always transported by waves of finite speed and only the discrete information $\mathcal{S}$ may be transported by waves of speed negative infinity. Initial data of type $U_l = (1, \mathcal{C})$ and $U_r = (1, \mathcal{F})$ or $U_l = (1, \mathcal{F})$ and $U_r = (1, \mathcal{C})$ will therefore immediately give rise to the solution $U(x,t) = (1, \mathcal{F})$ or $U(x,t) = (1, \mathcal{C})$, respectively. A sketch of the wave curve in phase space $(\rho, \mathcal{S})$ is given in Figure 1 and 2, respectively.

Proof. We distinguish the following different cases.

a) $U_l = U_r$.

The solution is the constant $U(x,t) = U_l$ and $TV(U(\cdot,t)) = TV(U_0) = 0$.

b) $\mathcal{S}_l = \mathcal{S}_r = \mathcal{F}$.

This problem is a standard Riemann problem for the smooth, monotone flux function $\rho \to F(\rho, \mathcal{F})$ with initial data $\rho_0(x) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases}$. Under the Lax-Entropy condition there is a unique solution $\rho(x,t)$ comprising of a Lax–shock if $\rho_l < \rho_r$ and a rarefaction else. Due to the monotonicity of F all waves have positive speed. Further, $\mathcal{S}(x,t) = \mathcal{F}$, $TV(U(\cdot,t)) \leq TV(U_0)$ and $\rho(x,t) \leq \max\{\rho_l, \rho_r\}$.

c) $U_l = (\rho_l, \mathcal{F})$ and $U_r = (1, \mathcal{C})$

Assume first $0 < \rho_l < 1$. Then, the solution consists of a shock wave of negative speed $s = -\frac{F(\rho_l, \mathcal{F})}{1-\rho_l}$ connecting $(\rho_l, \mathcal{F})$ and $(1, \mathcal{C})$. If $\rho_l = 0$ then the solution consists of a shock wave of zero speed. If $\rho_l = 1$, then the solution consists of a wave of speed negative infinity in $\mathcal{S}$ leading to $U(x,t) = (1, \mathcal{C}), t > 0$. In all cases $TV(U(\cdot,t))$ is either constant or zero and the maximum of $\rho(x,t)$ is one.

d) $U_l = (1, \mathcal{C})$ and $U_r = (\rho_r, \mathcal{F})$

If $\rho_r = 1$ then the solution consists of a wave of speed negative infinity in $\mathcal{S}$ and $U(x,t) = (1, \mathcal{F})$ for $t > 0$. The TV norm is decreasing and the maximum of $\rho(x,t)$ is one. Let $0 \leq \rho_r < 1$. The solution consists of a superposition of a wave of speed negative infinity connecting the left state $(1, \mathcal{C})$ to $(1, \mathcal{F})$ and a second wave of positive speed. The later is obtained by solving a standard Riemann problem as in case (c) with left datum $(1, \mathcal{F})$ and right datum $(\rho_r, \mathcal{F})$. The total variation of $U(x,t)$ at any time $t > 0$ is bounded by that of $U_0(x)$ and the maximum of $\rho(x, \cdot)$ is one.

■

Note that a wave of speed negative infinity arises if and only if $\rho_l = \rho_r = 1$ or if $U_l = (1, \mathcal{C})$. A wave of infinite speed only propagates information on $\mathcal{S}$, keeping the density ρ constant. Hence, we assume in the following that the initial data is well prepared (see Definition 2.1)) to avoid multiple infinte speed waves. If this is not fulfilled we observe at time $t = 0$ a series of waves of infinite speed flipping the variable $\mathcal{S}$ and the initial data is well prepared after this interaction. Two adjacent states at maximum density but different states $\mathcal{S}$ can not persist.

Definition 2.1. *The initial data* $U_0(x) = (\rho_0(x), \mathcal{S}_0(x)) : \mathbb{R} \to [0,1] \times \{\mathcal{F}, \mathcal{C}\}$ *with* $\rho_0 \in L^1(\mathbb{R})$ *is said to be well–prepared if the following implication holds. For every nontrivial interval* $J \subset \mathbb{R}$ *the following holds:*

$$\text{If } \rho_0(x) = 1 \text{ for all } x \in J, \text{ then } \mathcal{S}_0(x) = \mathcal{F} \text{ or } \mathcal{S}_0(x) = \mathcal{C} \text{ for all } x \in J. \tag{7}$$

Remark 2.1. *We offer the following interpretation of the solutions of Lemma 2.1, similar to* [2]*. Consider the following stepwise scenario: Initially, the density* ρ *is below the critical density* $\rho = 1$*, and the supply chain operates like a nonlinear transport process with velocity* $F(\rho, \mathcal{F})/\rho$*. Then, it is congested (possibly due to changing boundary conditions or machine failure) and say for* $x > 0$ *we have* $\mathcal{S} = \mathcal{C}$*. Then, necessarily* $\rho = 1$ *for* $x > 0$ *and a wave of finite negative speed arises moving backwards through the supply chain and transports the state* $(1, \mathcal{C})$*. This corresponds to filling up the buffers with the congested state. In a third phase assume that the supply chain may reach the free flow phase at some point again. Then, this information travels at infinite speed backwards through the supply chain changing all states previously being* $(1, \mathcal{C})$ *to* $(1, \mathcal{F})$*. This is precisely observed in* [2] *and illustrated below in Section 3 as numerical result.*

Considering the state constraint (1 b), we focus on the restricted state space $\{(\rho, \mathcal{F}) : 0 \leq \rho \leq 1\} \cup \{(1, \mathcal{C})\}$. Formally, the waves along which the value of $\mathcal{S}$ changes are considered of first family and the other of second family (standard Lax waves). This is due to the fact that the former waves have always negative speed (possibly $-\infty$), while the latter waves have positive speed. The possible reachable states and relative waves are shown in Figure 1 and 2.

Note that in Figure 1 the left state $U_l = (1, \mathcal{C})$ can be only connected to the right state $(1, \mathcal{F})$ by a wave of speed negative infinity transporting only the state information $\mathcal{S}$. This wave also appears if the left state is $(1, \mathcal{F})$ and if the right state is $U_r = (1, \mathcal{C})$, see also Figure 2. A zero speed in the first family occurs if and only if $U_l = (0, \mathcal{F})$ and

$U_r = (1, \mathcal{C})$ since $F(0, \mathcal{F}) = F(1, \mathcal{C}) = 0$. Finally, due to the monotonicity of F as a function of ρ the waves of the second family have non–negative speed.

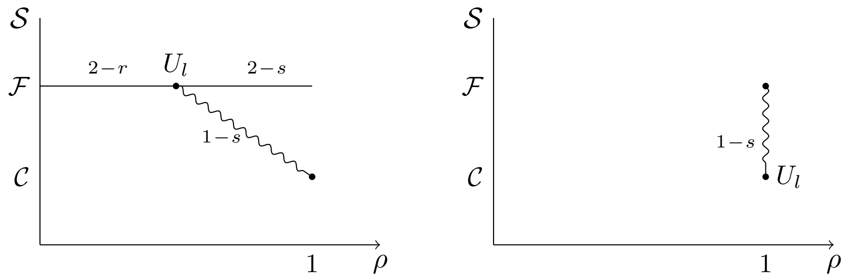

FIGURE 1. Reachable points for a given left state $U_l = (\rho, \mathcal{S})$ with $\mathcal{S} = \mathcal{F}$ (left) and $\mathcal{S} = \mathcal{C}$ (right). Here $2 - r$ indicates states connected by rarefaction waves of the second family, $2 - s$ by shock waves of the second family and $1 - s$ by shock waves of fist family. The 2 waves have always positive speed, while the 1 waves always negative speed (possibly equal to $-\infty$).

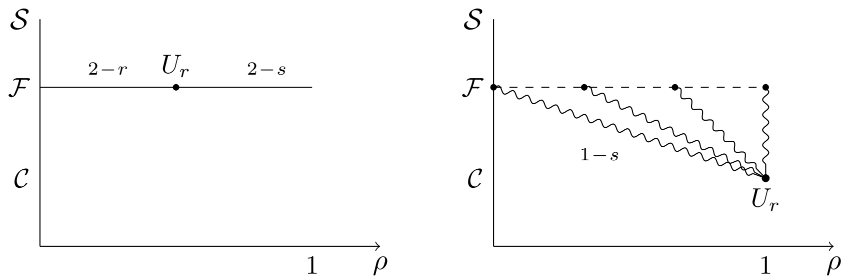

FIGURE 2. Reachable points for a given right state $U_r = (\rho, \mathcal{S})$ with $\mathcal{S} = \mathcal{F}$ (left) and $\mathcal{S} = \mathcal{C}$ (right). The legend is the same as Figure 1

Next, we turn to the proof of existence of solutions to Cauchy problems, by using wave front–tracking [4, 11]. Let $\rho_0(x) \in L^1(\mathbb{R})$, $\rho_0(x) \in [0, 1]$, and $\mathcal{S}_0(x)$ such that $TV(U_0) < \infty$ be given. Further, assume that $U_0 = (\rho_0, S_0)$ is well–prepared and that $\mathcal{S}_0(x) = \mathcal{C}$ implies $\rho_0(x) = 1$. Note that $\mathcal{S}_0$ can have only a finite number of jumps and is therefore already piecewise constant. Then, for $\nu \in \mathbb{N}$, we choose a sequence $U_{\nu,0} = (\rho_{\nu,0}, \mathcal{S}_{\nu,0})$ piecewise constant in ρ and $\mathcal{S}$ and such that $\mathcal{S}_{\nu,0}(x) = \mathcal{C}$ implies $\rho_{\nu,0}(x) = 1$. $(U_{\nu,0})_\nu$ can be chosen such that

$$TV(U_{\nu,0}) \leq TV(U_0), \quad \|\rho_{\nu,0}\|_\infty \leq \|\rho_0\|_\infty,$$
$$\|\rho_{\nu,0} - \rho_0\|_{L^1(\mathbb{R})} \leq \frac{1}{\nu} \text{ and } U_{\nu,0} \text{ is well–prepared.}$$

Fix $\nu \in \mathbb{N}$. Due to Definition 2.1 no waves of infinite speed arise initially. $U_{\nu,0}$ has a finite number of discontinuities $x_1 < x_2 < \cdots < x_N$ in at least one component. For each i we solve the Riemann problem generated by the initial data $(U_{\nu,0}(x_i)-, U_{\nu,0}(x_i)+)$

according to Lemma 2.1. If the solution contains only shocks then we use the exact solution, while for a rarefaction wave we split it in a centered rarefaction fan of a sequence of jumps of size at most $\frac{1}{\nu}$ traveling with a speed between the characteristic speeds of the states connected. This leads to a piecewise constant approximate solution $U_\nu(x,t)$ with traveling jump discontinuities. The solution exists until a time t_1 where at least two wave fronts interact. At time t_1 we repeat the construction until a second interaction time t_2 and so on. In order to prove well–posedness we need to estimate the number of waves and the number of interactions. Due to Lemma 2.1 the total variation does not increase at each interaction also after splitting the rarefaction wave into centered waves of size of at most $\frac{1}{\nu}$.

Lemma 2.2. *The number of wave fronts for the approximate solution U_ν is not increasing.*

Proof. Notice that, since $U_{\nu,0}$ is well prepared, only waves with finite speed are present in the approximate solution U_ν. Consider two interacting waves (U_l, U_m) and (U_m, U_r). Notice that a wave can not connect two congested states, otherwise it would be trivial.
If $U_l = U_m = U_r = \mathcal{F}$, then the assertion of the Lemma follows from standard results see for example [11, Lemma 2.6.4]. In particular the number of waves decreses by one.
If $U_l = \mathcal{C}$ then $U_m = \mathcal{F}$, but then this case is excluded because (U_l, U_m) would have speed $-\infty$. Similarly we exclude the case $U_l = U_r = \mathcal{F}$ and $U_m = \mathcal{C}$.
Finally the case $U_l = U_m = \mathcal{F}$ and $U_r = \mathcal{C}$ will give rise to the case (c) of Lemma 2.1 and the number of waves stays constant after the interaction. This concludes the proof. ∎

Lemma 2.3. *The number of interactions between waves is bounded.*

Proof. The number of interactions is bounded if all states belong have state $\mathcal{S} = \mathcal{F}$ due to standard results on wave-front tracking of scalar conservation laws, see [11, Lemma 2.6.4]. Hence, the total number of interactions of fronts moving at positive speed is bounded.

The state $(1, \mathcal{C})$ can not occur due to interactions of states with $\mathcal{S} = \mathcal{F}$. Hence, if initially $U_{\nu,0}$ contains the state $(1, \mathcal{C})$, then according to Lemma 2.1 this state propagates with non–positive speed. Every front moving at non–negative speed and interacting with this state give rise to wave of non–positive speed. Hence, every initial state $(1, \mathcal{C})$ can interact at most once with every front moving at positive speed. The number of initial states $(1, \mathcal{C})$ is bounded as well as the total number of wave fronts. Therefore, the total number of interactions is bounded. ∎

Finally, the wave front tracking approximation $t \to \rho_\nu(\cdot, t)$ is uniformly Lipschitz continuous with values in $L^1(\mathbb{R})$. If the state $(1, \mathcal{C})$ is not present in U_ν then the results follows for example due to [11, Theorem 2.6.6]. If U_ν contains the state $(1, \mathcal{C})$ a wave of finite negative speed is present and we obtain again uniform Lipschitz continuity. Due to Lemma 2.1 we have $\rho_\nu(x,t) \in [0,1]$.

Using the previous results we obtain the following result.

Theorem 2.1. *Consider equation (4) with initial condition $U_0(x) = (\rho_0(x), \mathcal{S}_0(x))$, such that $\rho_0(x) \in L^1(\mathbb{R}; [0,1])$ and $TV(U_0)$ is bounded. Assume F fulfills (3) and U_0 is well-prepared in the sense of Definition 2.1. Then, there exists a weak solution $U(x,t)$ to (4) defined for every $t \geq 0$.*

Proof. Note that since $TV(U_0)$ is bounded, $\mathcal{S}_0(x)$ has only a finite number of discontinuities. Consider now the sequence of wave front tracking approximations $U_\nu(x,t)$ constructed as described above. We have $TV(U_\nu(x,t)) \leq TV(U_0)$ and ρ_ν is uniformly Lipschitz. Due to Helly's theorem $\rho_\nu(x,t) \to \rho(x,t)$ in $L^1(\mathbb{R}\times\mathbb{R}_+,[0,1])$ and $\rho_\nu(x,0) \to \rho_0(x)$ in L^1 by construction. Since $\mathcal{S}_0(x)$ is piecewise constant we have for ν sufficiently large $\mathcal{S}_{\nu,0}(x) \equiv \mathcal{S}_0(x)$. As in [11, Theorem 2.6.6] we observe that $(\rho,\mathcal{S})$ is a weak solution to equation 4(a). ■

2.1. **Production line.** The previous dynamics can easily be extended to the case of a production line. A production line is a directed graph with vertices of degree equel to two (with one incoming arc and one outgoing arc.) We assume that every arc $i = 1,\dots,N$ is parameterized by an interval $x \in [a_i, b_i]$ where possibly $a_i = -\infty$ or $b_i = \infty$. The state on arc i is denoted by $U^i = (\rho^i, \mathcal{S}^i)$. Even if the flux function F might depend on i we discuss for simplicity the case where on every arc i, U^i is supposed to fulfill (4). At a vertex of degree two we need to couple the dynamics i and $i+1$ linking the boundary values $U_i(b_i,t)$ and $U_{i+1}(a_{i+1},t)$. In order to obtain a consistent solution to the Cauchy problem we prescribe the following coupling condition:

$$F(\rho_i(b_i,t), \mathcal{S}_i(b_i,t)) = F(\rho_{i+1}(a_{i+1},t), \mathcal{S}_{i+1}(a_{i+1},t)),\ t > 0. \tag{8}$$

The condition is sufficient to obtain a solution to the Riemann problem at the node as the following Lemma shows.

Lemma 2.4. *Assume a single vertex with two connected arcs i and $i+1$ such that $a_i = -\infty$, $b_i = a_{i+1}$ and $b_{i+1} = \infty$. Then, for any constant initial data $U_{k,0}$ and $k \in \{i, i+1\}$, there exists a solution $U_k(x,t)$ to (4) and (8).*

Let $\tilde{U}$ be the solution to the Riemann problem

$$(4)\,\textit{and}\ \tilde{U}_0(x) = \begin{cases} U_{i,0} & x < b_i = a_{i+1} \\ U_{i+1,0} & x > a_{i+1} = b_i \end{cases}. \tag{9}$$

Then, the solution $U_i(x,t)$ coincides with the restriction of $\tilde{U}$ for all $x < b_i$ and $U_{i+1}(x,t)$ coincides with the restriction of $\tilde{U}$ for all $x > a_{i+1}$.

Proof. We construct a solution to the network problem (4) and (8) by solving two (half-) Riemann problems. In order to obtain a solution for arc i we solve the Riemann problem for $U(x,t)$ on $\mathbb{R}$

$$(4)\text{and}\ U(x.0) = \begin{cases} U_{i,0} & x < b_i \\ U_i^+ & x > b_{i+1} \end{cases}. \tag{10}$$

for a state U_+^i defined below and set $U_i(x,t) = U(x,t)$ for $x \leq b_i$. Similarly, for arc $i+1$ we solve the Riemann problem for $U(x,t)$ on $\mathbb{R}$

$$(4)\text{and}\ U(x.0) = \begin{cases} U_{i+1}^+ & x < a_{i+1} \\ U_{i+1,0} & x > a_{i+1} \end{cases}. \tag{11}$$

for a state U_{i+1}^+ defined below and set $U_{i+1}(x,t) = U(x,t)$ for $x \geq a_{i+1}$. The states U_k^+ depend on the initial data as follows.

- $U_{i,0} = (\rho_{i,0}, \mathcal{F}_{i,0})$ and $U_{i+1,0} = (\rho_{i+1,0}, \mathcal{F}_{i+1,0})$.
  Then, $U_i^+ = (\rho_{i,0}, \mathcal{F}_{i,0}) = U_{i+1}^+$ and the solution to (10) and (11) is constant and consists of a wave of non–negative speed, respectively.

- $U_{i,0} = (\rho_{i,0}, \mathcal{F}_{i,0})$ and $U_{i+1,0} = (1, \mathcal{C}_{i+1,0})$.
  Then, $U_i^+ = (1, \mathcal{C}_{i+1,0}) = U_{i+1}^+$ and the solution to (10) and (11) is a shock of non–positive speed and constant, respectively. Note that there is the possibility of a shock of zero speed if $\rho_{i,0} = 0$ directly at the node. As in [13] we might modify the solution at this point of zero measure.
- $U_{i,0} = (1, \mathcal{C}_{i,0})$ and $U_{i+1,0} = (\rho_{i+1,0}, \mathcal{F}_{i+1,0})$.
  Then, $U_i^+ = (1, \mathcal{F}_{i,0}) = U_{i+1}^+$ and the solution to (10) and (11) is a wave of speed negative infinity changing immediately $\mathcal{C}$ to $\mathcal{F}$, i.e. $S_{i,0}^+ = \mathcal{F}$, and a wave of non–negative speed, respectively.

Due to construction condition (8) holds for $t > 0$. ∎

Remark 2.2. *In the last and second to last case the flux is zero at the node. Clearly, the construction is compatble with that of Lemma 2.1.*

Remark 2.3. *In case of different flux function $F_i \neq F_{i+1}$, we need to ensure, that (8) can be fulfilled. Therefore a necessary assumption is $F_i(1-) < F_{i+1}(1-)$.*

Summarizing, the problem of a production line yields the same solutions as the standard Riemann problem (6) if F is the same on the both connected arcs. Note that other coupling conditions will lead to different solutions, however, (8) is necessary to conserve the total mass. Also, (8) is obtained naturally by integration by parts on (4) and is equivalent to Kirchoff's law at the node. Since the dynamics is mainly governed by the monotone flux function $\rho \to F(\rho, \mathcal{S})$ one condition is sufficient, contrary to the case of vehicular traffic flow with strictly concave flux function, see [11].

Performing an analysis similar to the previous section, we obtain the following:

Theorem 2.2. *Consider equation (4) on a production line, with initial conditions $U_{i,0}(x) = (\rho_{i,0}(x), \mathcal{S}_{i,0}(x))$, such that $\rho_{i,0}(x) \in L^1([a_i, b_i]; [0,1])$ and $TV(U_{i,0})$ is bounded. Assume F fulfills (3) and $U_{0,i}$ is well-prepared in the sense of Definition 2.1. Then, there exists a weak solution $U_i(x,t)$ to (4) defined on the whole production line for every $t \geq 0$.*

Proof. The proof is similar to that of Theorem 2.1. The only difference is that, due to dynamics at nodes, infinite speed waves may be generated. However, the latter interact instantaneously with all waves in the incoming arc, without increasing the number of waves or the TV. Therefore the same arguments apply. ∎

3. Numerical illustrations

In order to show the different state of the systems we use a Godunov–method on an equidistant grid explained in more detail below. The computational domain is $(t,x) \in [0,1]^2$ discretized by $N_x = 100$ points in space. The function $\mathbf{F}$ is chosen as in [2, 3]:

$$\mathbf{F}(\rho) := \begin{cases} c\,(1 - \exp(-\rho)) & 0 \leq \rho < 1 \\ 0 & \rho = 1 \end{cases}, \tag{12}$$

with $c = \frac{1}{1-\exp(-0.8)}$, thus $\lim_{\rho \to 1^-} \mathbf{F}(\rho) = \frac{1-\exp(-1)}{1-\exp(-0.8)}$. As initial and boundary conditions we set

$$\rho(0,x) = 0.4 + 0.4(1-x)\cos(16\pi x), \mathcal{S}(0,x) = \mathcal{F} \text{ and } \rho(t,0) = 0.2, \mathcal{S}(t,0) = \mathcal{F}. \tag{13}$$

The boundary condition at $x = 1$ is always $\rho(t, 1) = 1$, but $\mathcal{S}(t, 1)$ changes over time switching from $\mathcal{F}$ to $\mathcal{C}$ at $t = 0.1$ and $t = 0.7$ and vice versa at $t = \frac{1}{2}$.

The time–discretization is according to the CFL conditions. Clearly, the derivative of the smooth part of $\mathbf{F}$ is bounded. Additionally, due to the maximum principle for ρ we may also bound a priori the wave speed of the possibly backwards moving wave connecting some $(\rho, F(\rho, \mathcal{F}))$ and $(1, F(1, \mathcal{C}) = 0)$. This determines the (global) time–step Δt which numerical value is seen as difference to $T = 1$ in the title of Figure 3.

Denote the discrete solution at time $n\Delta t$ and at point $i\Delta x$ by ρ_i^n and $\mathcal{S}_i^n$, respectively. In order to capture the dynamics introduced in Section 2 we proceed at every time–step as follows: We first solve for possibly waves of infinite speed by considering at each time step n all states with $\rho_i^n = 1$. Depending on the state $\mathcal{S}$ and density of a neighboring state $\mathcal{S}_{i-1}^n$ we possibly change the state $\mathcal{S}_{i-1}^n$. Note that such change may only occur if for any i $\rho_i^n = \rho_{i-1}^n = 1$ and $\mathcal{S}_i^n \neq \mathcal{S}_{i-1}^n$. Since waves have negative infinite speed, we proceed backward on index i.
In the second step we apply a Godunov solver based on the Riemann problems of Section 2 to obtain ρ_i^{n+1} and $\mathcal{S}_i^{n+1}$. Finally, we solve the equation (4b) by considering all cells i such that $\mathcal{S}_i^{n+1} = \mathcal{C}$ and set $\rho_i^{n+1} = 1$.

The oscillations in the initial data of ρ are transported as seen in Figure 3 followed by a shock wave of positive speed. In Figure 3 and 4 we show the evolution of ρ and the phases of the system with white color corresponding to free flow and black color corresponding to congested flow. The horizontal line at $t = \frac{1}{2}$ in 0 4 corresponds to the wave of infinite speed traveling through the system as soon as the boundary condition changes from congested to free flow. In the density (Figure 3) we observe the corresponding slow forward moving shock wave between time $t = \frac{1}{2}$ and $t = 0.7$, when the boundary condition once again changes. Due to numerical dissipation this forward moving shock wave spreads over about ten grid cells. In Figure 4 we observe backwards traveling information (of the congested phase) starting at $t = 0.1$ and $t = 0.7$, respectively. The wave speed depends on the current density in the production line and is different in both cases. The different speed of propagation is also clearly seen in the corresponding part of Figure 3.

4. Summary

We discussed some analytical properties for a recently introduced supply chain model with discontinuous flux function. After suitable reformulation an analytical treatment as well as a numerical procedure has been applied to obtain an existence result, computational results and the extension towards supply chain networks.

Acknowledgments

This work has been supported by HE5386/7-1 and DAAD 54365630.

References

[1] D. Armbruster, P. Degond, and C. Ringhofer, *Kinetic and fluid models for supply chains supporting policy attributes*, Bull. Inst. Math. Acad. Sin. (N.S.), 2 (2007), pp. 433–460.

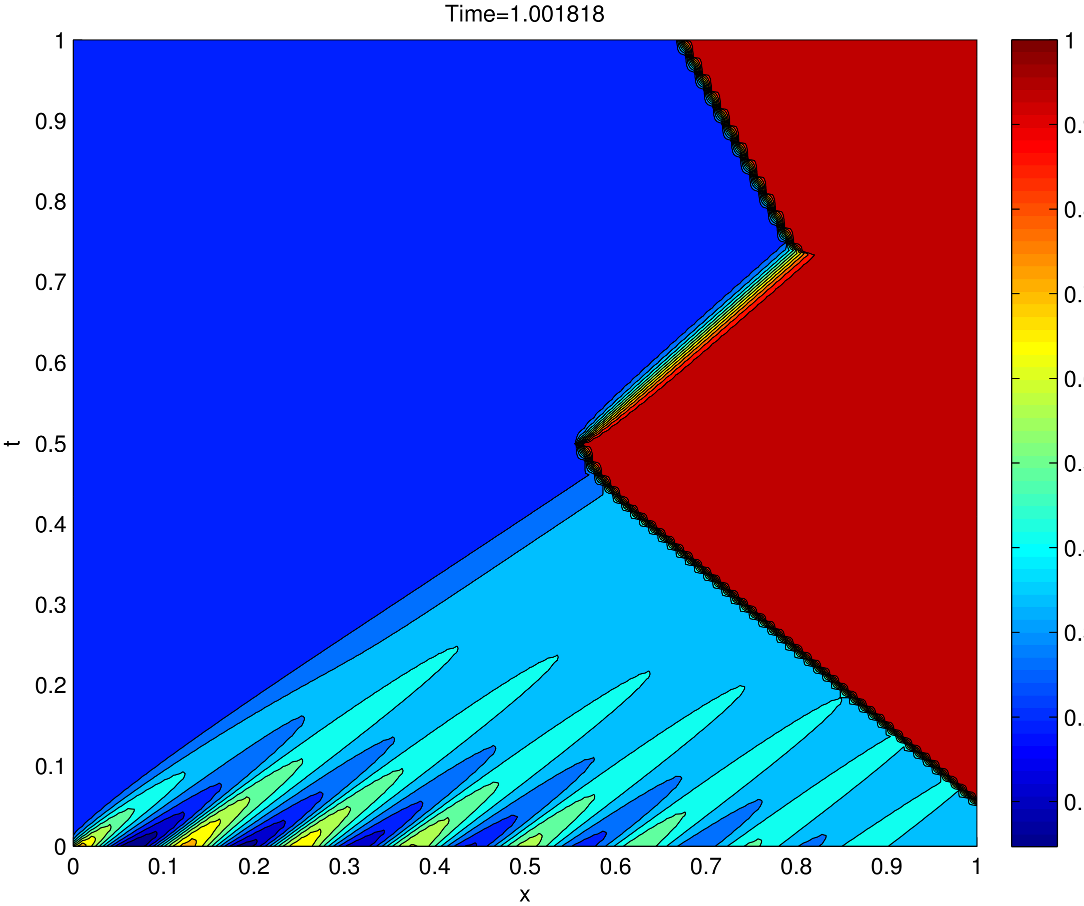

FIGURE 3. Spatial and temporal evolution of the product density for initial and boundary data (13) and prescribed time–dependent phases and maximal density along the boundary $x = 1$. Shown are the level curves of ρ. Terminal time is indicated in the title showing the relatively small time–step as difference to $T = 1$.

[2] D. ARMBRUSTER, S. GÖTTLICH, AND M. HERTY, *A continuous model for supply chains with finite buffers*, SIAM J. Appl. Math., (2011).

[3] D. ARMBRUSTER, D. MARTHALER, AND C. RINGHOFER, *Kinetic and fluid model hierarchies for supply chains*, Operations Research, 2 (2003), pp. 43–61.

[4] A. BRESSAN, *Hyperbolic systems of conservation laws*, vol. 20 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[5] C. D'APICE, S. GÖTTLICH, M. HERTY, AND B. PICCOLI, *Modeling, simulation, and optimization of supply chains*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2010. A continuous approach.

[6] C. D'APICE, R. MANZO, AND B. PICCOLI, *Existence of solutions to Cauchy problems for a mixed continuum-discrete model for supply chains and networks*, J. Math. Anal. Appl., 362 (2010), pp. 374–386.

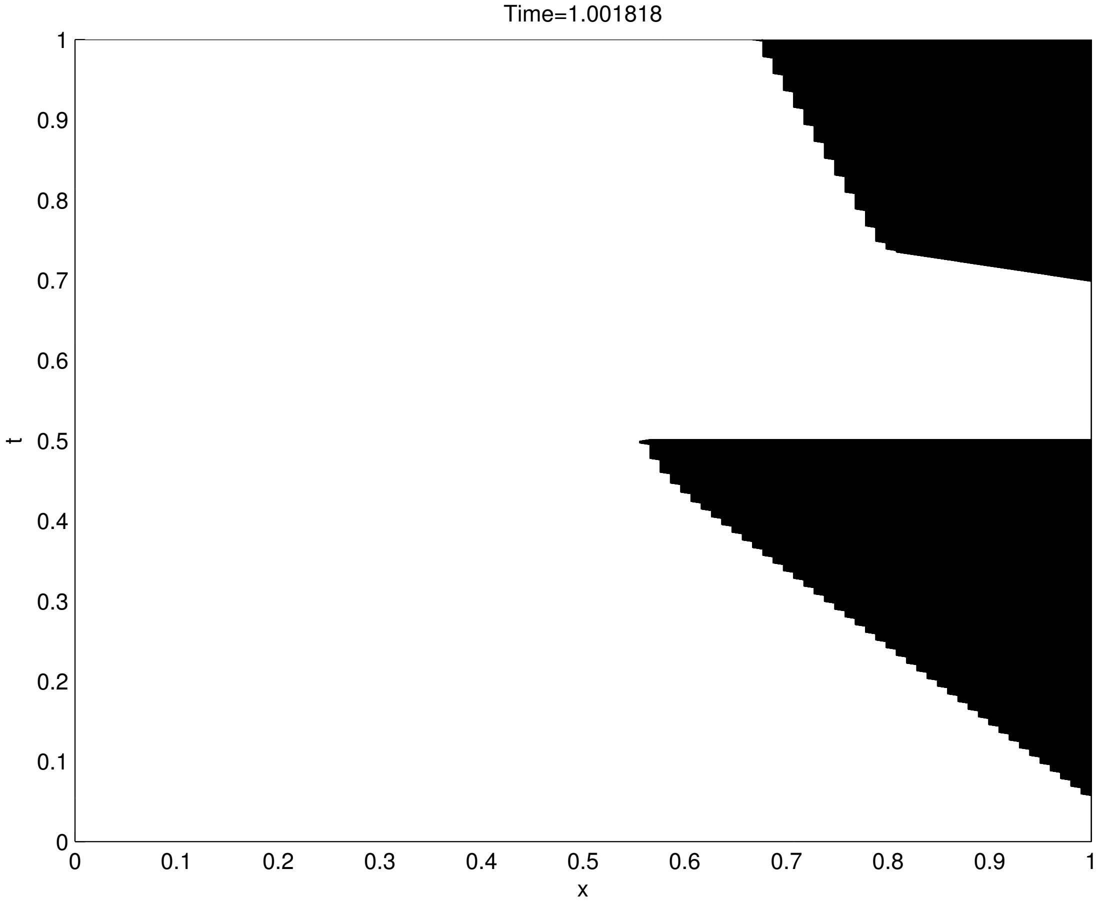

FIGURE 4. Spatial and temporal evolution of the phase for initial and boundary data (13). White and black colors correspond to free and congested phases. The boundary condition at $x = 1$ changes phase at predefined times. Terminal time is indicated in the title showing the relatively small time–step as difference to $T = 1$.

[7] J.-P. Dias and M. Figueira, *On the approxmation of the solutions of the riemann problem for a discontinuous conservation law*, Bull. Braz. Math. Soc. (N.S.), 36 (2005), pp. 115–125.

[8] ———, *On the viscous cauchy problem and the existence of shock profiles for a p–system with a discontinuous stress function*, Quart. Appl. Math., 63 (2005), pp. 335–341.

[9] J.-P. Dias, M. Figueira, and J.-F. Rodrigues, *Solutions to a scalar discontinuous conservation law in a limit case of phase transitions*, Journal of Mathematical Fluid Mechanics, (2005), pp. 153–163.

[10] M. Garavello, R. Natalini, B. Piccoli, and A. Terracina, *Conservation laws with discontinuous flux*, Networks and Heterogenous Media, 2 (2007), pp. 159–179.

[11] M. Garavello and B. Piccoli, *Traffic flow on a road network using the Aw-Rascle model*, Comm. Partial Diff. Equ., 31 (2006), pp. 243–275.

[12] S. Göttlich, M. Herty, and A. Klar, *Modelling and optimization of supply chains on complex networks*, Commun. Math. Sci., 4 (2006), pp. 315–330.

[13] H. Holden and N. H. Risebro, *A mathematical model of traffic flow on a network of unidirectional roads*, SIAM J. Math. Anal., 26 (1995), pp. 999–1017.
[14] M. La Marca, D. Armbruster, M. Herty, and C. Ringhofer, *Control of continuum models of production systems*, IEEE Transactions on automatic control, to appear (2010).
[15] J. Wiens, J. Stockie, and J. Williams, *Riemann solver for a kinematic wave traffic model with a discontinuous flux*, p. 29.

RWTH Aachen University, Templergraben 55, D-52056 Aachen, Germany.

RWTH Aachen University, Templergraben 55, D-52056 Aachen, Germany.

Rutgers University - Camden, 227 Penn Street, Camden, NJ 08102, USA.