MOVING BOTTLENECKS IN CAR TRAFFIC FLOW: A PDE-ODE COUPLED MODEL

CORRADO LATTANZIO *, AMELIO MAURIZI †, AND BENEDETTO PICCOLI ‡

Abstract. We study a model of vehicular traffic flow represented by a coupled system formed by a scalar conservation law, describing the evolution of cars density, and an ODE, whose solution is the position of a *moving bottleneck*, i.e. a slower vehicle moving inside the cars flow. A fractional step approach is used to approximate the coupled model and convergence is proved by compactness arguments. Finally, the limit of such approximating sequence is proved to solve the original PDE–ODE model.

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1. Introduction. Vehicular traffic flow is a complex phenomenon, which has attracted the interest of researchers from many fields (engineering, physics and math) and was studied using many different approaches and scales (from micro to macroscopic). In particular, the fluid dynamic paradigm, started in 50s by Lighthill, Whitham and Richards (see [18, 19]), was strongly developed also in recent years. On one side, researchers proposed more accurate models, for istance [2, 8], on the other side extensions to networks was achieved, see [7, 15].

It is known that vehicular traffic is composed by vehicles with different characteristics, such as motorcycles, cars, buses, trucks etc. For this reason, various multipopulation models were also considered, see [3, 21].

Also researchers addressed the problem of detecting the trajectory of a single car in the whole traffic flow. In [10, 11], authors considered a PDE–ODE model, where the LWR model, consisting of a single conservation law, is used for the traffic flow evolution and an ODE for the position of the single car. In this case the car is assumed not to influence significantly the traffic flow, thus the system is not fully coupled: One may first solve the PDE and then the ODE. The relative numerics was developed in [6], where also the convergence rate was studied. Notice that the system is multiscale, in the sense that both the macroscopic car density and the microscopic position of a single car are modelled.

Here we consider the more complicated situation in which the position of the single vehicle influences the whole traffic flow, thus giving rise to a fully coupled micro-macro model. The situation is that of moving bottlenecks, where a large and slow vehicle, such as a bus or a truck, is producing a non negligible capacity dropping of the vehicular flow.

The problem of modelling bottlenecks, moving or not, have been addressed first by the engineering community, then also by the applied math ones for the static case. Examples of bottlenecks modelling are [9] and [14]. In the first paper, the authors considered the case of a toll gate giving rise to unilateral pointwise constraint on the maximal flux. While the second paper considers the case of reduction of road physical

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dimensions (for istance because of works) modelled by a simple junctions with one incoming and one outgoing roads of different capacity. In both papers, the Cauchy problem is proved to be well-posed.

We shall prove existence of solutions by means of fractional step method: in successive time steps, we first solve the conservation law with the slower vehicle position fixed and then the ODE using the car density given by the first step. This removes the coupling among the two equations, thus we can take advantage of the results for the uncoupled case. Solutions to the ODE are to be intended in the sense of Filippov [13].

To obtain compactness for the fractional step approximating sequence is more standard, while the major difficulties are in proving that the limit solves the coupled system. The uniform convergence of the slower vehicle trajectories permits to pass in the limit in the integral formulation of the PDE. On the other side, we need to study carefully the weak convergence of the density derivative as a measure to pass to the limit in the differential inclusions corresponding to the ODE. In particular, we use recent results by Bressan and LeFloch on the structure of shocks in Wave Front Tracking approximations ([5]) and more standard achievements from the theory of differential inclusions ([1]).

This paper is organized as follows. In §2, we introduce the model and our notion of solution. The following section is devoted to the description of the fractional step approximation of our coupled model and to its convergence. In §4 we then prove that the limit solve the original coupled model.

2. Basic definitions. A slow moving large vehicle along a street reduces the capacity along the road and thus it generates a *moving bottleneck* for the car traffic flow. Moreover, its dynamics can be described by an evolution taking into account the total density of cars. Following these considerations, and focusing on the LWR model for car traffic flow, we reach the following mathematical formulation:

$$\begin{cases} \rho_t + f(x, y(t), \rho)_x = 0 \\ \rho(0, x) = \rho_0(x) \\ \dot{y}(t) = w(\rho(t, y)) \\ y(0) = y_0, \end{cases} \tag{2.1}$$

where $\rho = \rho(t, x) \in [0, \rho_{\max}]$ is the density of cars, $y = y(t)$ is the position of the slower vehicle and $w = w(\rho)$ is its speed, and the flux function f is given by

$$f(x, y(t), \rho) = \rho \cdot v(\rho) \cdot \varphi(x - y(t)). \tag{2.2}$$

In (2.2), $\varphi(\xi)$ is a function representing the capacity dropping of car flows, due to the presence of the slower vehicle and, as it is costumary for traffic flow models, the velocity $v(\rho)$ is assumed to satisfy

(i) $v : [0, \rho_{\max}] \to [0, v_{\max}]$;
(ii) $v(\rho_{\max}) = 0$ and $v(0) = v_{\max}$;
(iii) $v(\rho)$ is smooth and decreasing;
(iv) $\dfrac{d^2(\rho v)}{d\rho^2} < 0$.

We assume further the cut–off function $\varphi(\xi)$ to be smooth and equal to one outside a compact set containing the origin.

EXAMPLE 2.1. Let $\varphi(\xi)$ satisfy

1. $\beta \leq |\varphi(\xi)| \leq 1$, for any $\xi \in \mathbb{R}$;
2. $\varphi(\xi) = 1$ in $(-\infty, -2\delta) \cup (2\delta, +\infty)$;
3. $\varphi(\xi) = \beta$ in $(-\delta, \delta)$;
4. $\varphi(\xi)$ decrease on $(-2\delta, -\delta)$ and increase on $(\delta, 2\delta)$;

for given $0 < \beta < 1$ and $\delta > 0$, see Figure 2.1 for an example of $\varphi(\xi)$ satisfying (i)-(iv).

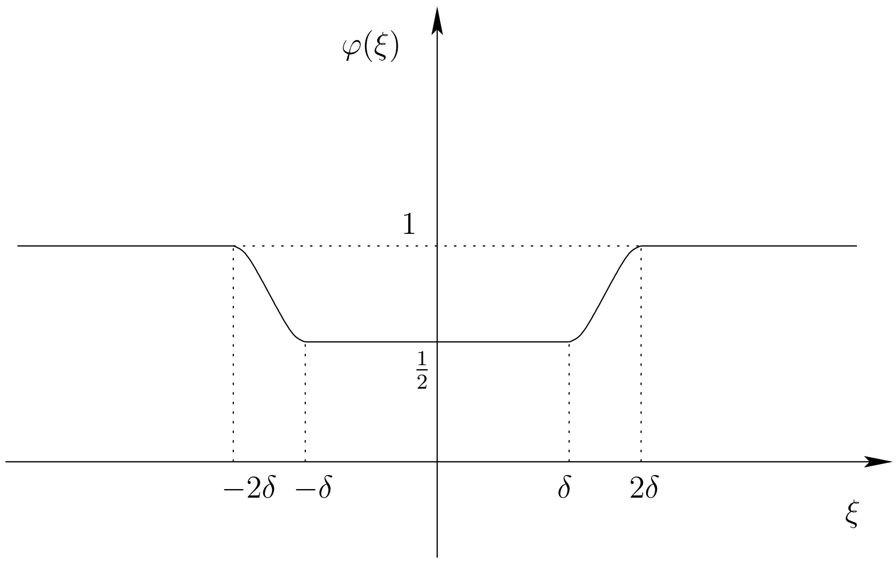

FIG. 2.1. *An example of the cut-off function $\varphi(\xi)$ with $\beta = \frac{1}{2}$*

The main purpose of the present paper is the proof of the existence of a solution (ρ, y) for density of cars and position of the slower moving vehicle to (2.1), for a suitable concept of solution, with initial datum $\rho_0 \in \mathcal{BV}$.

Throughout the paper, when stating $\rho \in \mathcal{BV}$ we mean the following $\mathcal{BV}$ norm is bounded:

$$\|\rho\|_{\mathcal{BV}} = \|\rho\|_{L_1} + TV(\rho),$$

where $TV(\rho)$ stands for the total variation of ρ.

Therefore, the condition $\rho_0 \in \mathcal{BV}$ implies in particular ρ_0 is in L_∞ and therefore we can rescale the equation to have $\rho_{\max} = v_{\max} = 1$, $0 \leq \rho_0(x) \leq 1$.

Our notion of solution is stated in the following definition.

DEFINITION 2.2. *A vector (ρ, y) is a solution to (2.1) in $[0, T]$ if $\rho(t, \cdot)$ is a function in $\mathcal{BV}(\mathbb{R})$ for a.e. $t \in [0, T]$ which solves the PDE in the sense of distributions, that is*

$$\int_0^T \int_{\mathbb{R}} \{\rho\phi_t(t, x) + f(x, y(t), \rho)\phi_x(t, x)\} dx dt + \int_{\mathbb{R}} \rho_0 \phi(0, x) dx = 0,$$

for any differentiable function $\phi(t, x)$ with compact support in $t \geq 0$.

Moreover, the position of the slower moving vehicle $y(t)$ solves the ODE in $[0, T]$ in the sense of Filippov, namely $y(t)$ is an absolutely continuous function such that

$$\dot{y} \in \overline{\text{co}}\{w(\rho) : \rho \in \mathcal{I}[\rho(t, y(t)-), \rho(t, y(t)+)]\}$$

for a.e. $t \in [0, T]$*, where the set* $\mathcal{I}[a, b]$ *is defined as the smallest interval containing* a *and* b*.*

3. Fractional step approach. In this section, we apply the fractional step approach to the coupled system (2.1) to construct approximations to its solution. We then prove the convergence of the related sequence by means of compactness arguments.

To this aim, for any fixed time $T > 0$, we define a time step $\triangle t$ and a number of iterations n such that $n\triangle t = T$. At first step, we obtain $\rho^1(t, x)$ starting from $\rho^0(t, x) = \rho_0(x)$ by solving in the time interval $(0, \triangle t]$

$$\begin{cases} \rho^1_t + f(x, y_0, \rho^1)_x = 0 \\ \rho^1(0, x) = \rho_0(x), \end{cases}$$

where y_0 is the initial datum of the ODE. Then we solve in $(0, \triangle t]$

$$\begin{cases} \dot{y}^1(t) = w(\rho^1(t, y^1(t))) \\ y^1(0) = y_0 \end{cases}$$

to obtain next term in the sequence for y. Iteratively, at the generic k-th step, the two decoupled Cauchy problems become:

$$\begin{cases} \rho^k_t + f(x, \bar{y}^{k-1}, \rho^k)_x = 0, \quad t \in (0, \triangle t] \\ \rho^k(0, x) = \rho^{k-1}(\triangle t, x) \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dot{y}^k(t) = w(\rho^k(t, y^k)), \quad t \in (0, \triangle t] \\ y^k(0) = y^{k-1}(\triangle t), \end{cases} \tag{3.2}$$

where $\bar{y}^{k-1} = y^{k-1}(\triangle t)$. In the following Sections 3.1 and 3.2 we shall obtain uniform bounds for the functions ρ^k and y^k, which will imply the convergence of the related approximating sequences in Section 3.3.

3.1. Scalar conservation law for car density. The densities ρ^k satisfy a scalar conservation law thus their properties are easily derived in the following proposition.

PROPOSITION 3.1. *Let* ρ^k *be the solution of*

$$\begin{cases} \rho^k_t + \left(\rho^k v(\rho^k)\varphi(x - \bar{y}^{k-1})\right)_x = 0 \\ \rho^k_0(x) = \rho^k(0, x) = \rho^{k-1}(\triangle t, x). \end{cases} \tag{3.3}$$

Then, if $\rho_0(x) \in \mathcal{BV}$ *with* $0 \le \rho_0(x) \le 1$*, for any* k *and for any* $t \in (0, \triangle t]$ *we have*

(i) $\|\rho^k(t, \cdot)\|_{L_1} \le \|\rho_0\|_{L_1}$

(ii) $0 \le \rho^k(t, x) \le 1$

(iii) $\|\rho^k(t, \cdot)\|_{\mathcal{BV}} \le e^{C_0 \triangle t} \|\rho^k_0(\cdot)\|_{\mathcal{BV}}$.

Proof.

(i) As it is well known [17], if ρ and $\bar{\rho}$ are two solutions of the general conservation law

$$\rho_t + f(t, x, \rho)_x = 0 \tag{3.4}$$

corresponding to L_1 initial data ρ_0 and $\bar{\rho}_0$ respectively, we get

$$\|\rho(t,\cdot)-\bar{\rho}(t,\cdot)\|_1 \le \|\rho_0-\bar{\rho}_0\|_1.$$

Due to the particular expression of our flux function $f(x,\rho)=\rho^k v(\rho^k)\varphi(x-\bar{y}^{k-1})$, if we choose $\bar{\rho}_0=0$ we obtain $\bar{\rho}=0$ and therefore we end up with

$$\begin{array}{ll}\|\rho^k(t,\cdot)\|_1 & \le \|\rho_0^k\|_1 = \|\rho^{k-1}(\triangle t,\cdot)\|_1 \\ & \le \|\rho_0^{k-1}\|_1 = \|\rho^{k-2}(\triangle t,\cdot)\|_1 \\ & \dots \\ & \le \|\rho_0\|_1 \end{array}$$

for any k.

(ii) The claim is trivially true for $k=0$, since $\rho^0(t,x)=\rho_0(x)$. For any k assume by induction $0\le\rho_0^k(x)=\rho^{k-1}(\triangle t,x)\le 1$. By comparison principle we easy conclude $0\le\rho^k(t,x)\le 1$ because $\rho\equiv 0$ and $\rho\equiv 1$ are solutions of (3.3) if $\rho_{\max}=1$, since $v(1)=0$ (speed of cars is zero at maximal density).

(iii) This result follows again from L_1-contraction property of scalar conservation laws. We start by considering the vanishing viscosity approximation of our conservation law (3.3) and we restrict ourselves to the case of smooth initial data $\rho_0(x)$:

$$\rho_t^{k,\varepsilon}+f(x,\bar{y}^{k-1},\rho^{k,\varepsilon})_x=\varepsilon\rho_{xx}^{k,\varepsilon}. \tag{3.5}$$

Thus, taking the derivative with respect to x in the above equation and denoting $z=\rho_x^{k,\varepsilon}$ we have

$$z_t+\big(f_\rho(x,\bar{y}^{k-1},\rho^{k,\varepsilon})z\big)_x+\big(f_x(x,\bar{y}^{k-1},\rho^{k,\varepsilon})\big)_x=\varepsilon z_{xx}. \tag{3.6}$$

As usual, we introduce a smooth approximation of the modulus function $\tau_\nu(\xi)$ as follows:

$$\tau_\nu(\xi)=\begin{cases}\xi^2/(4\nu) & |\xi|\le 2\nu\\ |\xi|-\nu & 2\nu<|\xi|\le+\infty.\end{cases}$$

Note that, as $\nu\downarrow 0$, we get pointwise,

$$\begin{aligned}&\tau_\nu(\xi)\to|\xi|\\&\tau_\nu'(\xi)\to\operatorname{sgn}(\xi)\\&\tau_\nu''(\xi)\xi\to 0.\end{aligned}$$

Multiplying on the left the equation (3.6) by $\tau_\nu'(z)$, since $\big(\tau_\nu'(z)f_\rho z\big)_x=\tau_\nu''(z)f_\rho z z_x+\tau_\nu'(z)\big(f_\rho z\big)_x$ we obtain

$$\begin{aligned}&\tau_\nu'(z)z_t+\Big(\tau_\nu'(z)f_\rho z\Big)_x-\tau_\nu''(z)f_\rho z_x z+\tau_\nu'(z)f_{xx}+\tau_\nu'(z)f_{x\rho}z\\&=\varepsilon\big(\tau_\nu'(z)z_x\big)_x-\varepsilon\tau_\nu''(z)(z_x)^2\end{aligned}$$

and since $\tau_\nu''(z)\ge 0$, integrating on x we end up with

$$\int_{-\infty}^{+\infty}[\tau_\nu'(z)z_t-\tau_\nu''(z)f_\rho z_x z+\tau_\nu'(z)f_{xx}+\tau_\nu'(z)f_{x\rho}z]\,dx\le 0.$$

Now, taking the limit for $\nu \downarrow 0$, we get

$$\int_{-\infty}^{+\infty} \Big[|z|_t + \operatorname{sgn}(z)\big(f_{xx} + z f_{x\rho}\big)\Big]\, dx \le 0.$$

Since $f_{xx}(x, \rho^{k,\varepsilon}, \bar{y}^{k-1}) = \varphi''(x - \bar{y}^{k-1})\rho^{k,\varepsilon} v(\rho^{k,\varepsilon})$ and $f_{x\rho}(x, \rho^{k,\varepsilon}, \bar{y}^{k-1}) = \varphi'(x - \bar{y}^{k-1})\big(\rho^{k,\varepsilon} v(\rho^{k,\varepsilon})\big)'$, we can rewrite the last inequality as follows:

$$\begin{aligned}
\int_{-\infty}^{+\infty} |z|_t\, dx &\le \int_{-\infty}^{+\infty} \sup_x \{\big|\varphi'(\cdot)\big(v(\rho^{k,\varepsilon}) + \rho^{k,\varepsilon} v'(\rho^{k,\varepsilon})\big)\big|\} \cdot |z|\, dx + \\
&+ \int_{-\infty}^{+\infty} \sup_x \{\big|\varphi''(\cdot) v(\rho^{k,\varepsilon})\big|\} \rho^{k,\varepsilon}\, dx \\
&\le \int_{-\infty}^{+\infty} C_1 |z|\, dx + \int_{-\infty}^{+\infty} C_2 |\rho^{k,\varepsilon}|\, dx,
\end{aligned}$$

where

$$C_1 = M_1 \sup_x \{\big|v(\rho^{k,\varepsilon}) + \rho^{k,\varepsilon} v'(\rho^{k,\varepsilon})\big|\}$$

$$C_2 = M_2 \sup_x \{\big|v(\rho^{k,\varepsilon})\big|\},$$

depend only on $\|\rho_0\|_\infty$ due to maximum principle, and because $|\varphi'| \le M_1$, $|\varphi''| \le M_2$. Since we also know

$$\int_{-\infty}^{\infty} |\rho^{k,\varepsilon}|_t\, dx \le 0$$

(this is another recasting of the L^1–contractivity for (viscous) conservation laws proved in point (i)), summing up the two inequalities, integrating in time and using Gronwall's Lemma we obtain

$$\int_{-\infty}^{+\infty} (|\rho^{k,\varepsilon}| + |\rho_x^{k,\varepsilon}|)\, dx \le e^{C_0 t} \int_{-\infty}^{+\infty} (|\rho_0^k| + |(\rho_0^k)_x|)\, dx \tag{3.7}$$

with $C_0 = \max\{C_1, C_2\}$. In (3.7), the integral term on the right hand side is the $\mathcal{BV}$–norm of the initial data, provided $\rho_0^k(x) \in C^1(\mathbb{R})$.
Assume now that the initial datum for (3.5) is only $\mathcal{BV}(\mathbb{R})$. Then we consider a mollified initial datum, obtained as a convolution of the previous one with a standard mollifier $\Phi^\mu(x) \in C^\infty(\mathbb{R})$:

$$\rho_{0,\mu}^k(x) = (\rho_0^k * \Phi^\mu)(x). \tag{3.8}$$

Hence,

$$\int_{-\infty}^{+\infty} (|\rho_{0,\mu}^k(x)| + |(\rho_{0,\mu}^k(x))_x|)\, dx = \|\rho_{0,\mu}^k(x)\|_{\mathcal{BV}} \le \|\rho_0^k(x)\|_{\mathcal{BV}}. \tag{3.9}$$

Then, for (3.7) we obtain

$$\|\rho_\mu^{k,\varepsilon}\|_{\mathcal{BV}} = \int_{-\infty}^{+\infty} (|\rho_\mu^{k,\varepsilon}| + |(\rho_\mu^{k,\varepsilon})_x|)\, dx \le e^{C_0 t} \int_{-\infty}^{+\infty} (|\rho_{0,\mu}^k| + |(\rho_{0,\mu}^k)_x|)\, dx$$

and, by (3.9),

$$\|\rho_\mu^{k,\varepsilon}\|_{\mathcal{BV}} \le e^{C_0 t}\|\rho_0^k\|_{\mathcal{BV}}.$$

Since

$$\rho_{0,\mu}^k(x) \longrightarrow \rho_0^k(x)$$

in L_1 as $\mu \to 0$, using again L_1 contractivity, we get

$$\|\rho_\mu^{k,\varepsilon}(t,\cdot) - \rho^{k,\varepsilon}(t,\cdot)\|_{L_1} \le \|\rho_{0,\mu}^k(\cdot) - \rho_0^k(\cdot)\|_{L_1} \longrightarrow 0,$$

as $\mu \to 0$ for any $t \in [0, \triangle t]$. Then

$$\|\rho^{k,\varepsilon}\|_{\mathcal{BV}} \le \liminf_{\mu} \ \|\rho_\mu^{k,\varepsilon}\|_{\mathcal{BV}} \le e^{C_0 t} \ \|\rho_0^k\|_{\mathcal{BV}}.$$

Using again the same contractivity property [17], we conclude $\rho^{k,\varepsilon} \to \rho^k$ in $L_1(\mathbb{R})$ as $\varepsilon \to 0$, thus, for any $t \in [0, \triangle t]$,

$$\|\rho^k\|_{\mathcal{BV}} \le \liminf_{\varepsilon} \ \|\rho^{k,\varepsilon}\|_{\mathcal{BV}} \le e^{C_0 t}\|\rho_0^k\|_{\mathcal{BV}} \le e^{C_0 \triangle t}\|\rho_0^k\|_{\mathcal{BV}}, \tag{3.10}$$

which concludes the proof.
□

3.2. Ordinary differential equation for slow moving vehicle trajectory. We turn now to the study of the ODE solved by the slow moving vehicle position $y^k(t)$. Since the right hand side of the equation $(2.1)_3$ is discontinuous, we need to consider it in the sense of Filippov [13]. By Theorem 1 in [10, 11], we know there exists an unique Filippov solution to (3.2) under the following assumptions:

1. mean speed of cars $v : [0, \rho_{\max}] \to [0, +\infty)$ is smooth, decreasing, with $v(\rho_{\max}) = 0$ and $\frac{d^2(\rho v(\rho))}{d\rho^2} < 0$.

2. speed of the slow moving vehicle $w : [0, \rho_{\max}] \to [0, +\infty)$ is smooth and decreasing.

3. there exists a constant $\alpha \in (0, 1)$ such that

$$\sup_{\rho \in (0,1]} \frac{w(0) - w(\rho)}{w(0) - f_\rho(x, y, \rho)} = \sup_{\rho \in (0,1]} \frac{w(0) - w(\rho)}{w(0) - (\rho v(\rho))'\varphi(x - y)} < 1 - \alpha. \tag{3.11}$$

In [10, 11], the authors provide sufficient conditions to satisfy hypothesis (3.11) without cut–off function φ, that are either

$$w(\rho) \ge v(\rho) \text{ together with } w(0) > v(0),$$

or $w(\rho) = v(\rho)$. However, these conditions are both not realistic for our case, because we need $w(\rho) \le v(\rho)$ (e.g. the vehicle is slower than cars) and we need to take into account the presence of the cut–off function. Let us assume $\beta \le \varphi(\xi) \le 1$ and

$$w(0) \ge \frac{d}{d\rho}(\rho v(\rho))\bigg|_{\rho=0},$$

(more general situations for smaller $w(0)$ can be also discussed, but we shall omit them here). Hence we obtain

$$\frac{w(0) - w(\rho)}{w(0) - (\rho v(\rho))'\varphi(x - y)} \le \frac{w(0) - w(\rho)}{w(0) - (\rho v(\rho))'}$$

for any ρ such that $(\rho v(\rho))' \geq 0$ and

$$\frac{w(0)-w(\rho)}{w(0)-(\rho v(\rho))'\varphi(x-y)} \leq \frac{w(0)-w(\rho)}{w(0)-\beta(\rho v(\rho))'}$$

for any ρ such that $(\rho v(\rho))' \leq 0$. Thus we obtain the following two sufficient conditions, similar to the one of [11], to be checked to obtain existence of an unique Filippov solution for the ODE:

$$\sup_{\rho\in(0,1],\ (\rho v(\rho))'\geq 0} \frac{w(0)-w(\rho)}{w(0)-(\rho v(\rho))'} < 1-\alpha; \tag{3.12}$$

$$\sup_{\rho\in(0,1],\ (\rho v(\rho))'\leq 0} \frac{w(0)-w(\rho)}{w(0)-\beta(\rho v(\rho))'} < 1-\alpha. \tag{3.13}$$

We shall solve the above conditions in the next specific example.

EXAMPLE 3.2. Assume $\rho_{max} = v_{max} = 1$ and $w(0) = 1$, that is the slow vehicle travels at maximal velocity when there are no cars along the road, which is meaningful for urban traffic and assume $\beta = 1/2$. Now, choosing a linear velocity $v(\rho) = 1-\rho$, conditions (3.12) and (3.13) read

$$1-w(\rho) < (1-\alpha)(1-(1-2\rho)) = 2(1-\alpha)\rho,\ \ \rho \leq \frac{1}{2}$$

and

$$1-w(\rho) < (1-\alpha)\left(1-\frac{1}{2}(1-2\rho)\right) = (1-\alpha)\left(\frac{1}{2}+\rho\right),\ \ \rho \geq \frac{1}{2}.$$

Thus,

$$w(\rho) > 1-2(1-\alpha)\rho,\ \ \rho \leq \frac{1}{2}$$

and

$$w(\rho) > 1-(1-\alpha)\left(\frac{1}{2}+\rho\right),\ \ \rho \geq \frac{1}{2}.$$

Now, in order to have $w(\rho) \leq v(\rho) = 1-\rho$ for any ρ under consideration, we need

$$1-2(1-\alpha)\rho < 1-\rho,\ \ \rho \leq \frac{1}{2}$$

and

$$1-(1-\alpha)\left(\frac{1}{2}+\rho\right) < 1-\rho,\ \ \rho \geq \frac{1}{2}.$$

The first inequality is fulfilled, provided $\alpha < 1/2$, while the second requires $\alpha < 1/3$. In conclusion, it is possible to verify (3.12), (3.13) for $0 < \alpha < 1/3$ and $w(0) = 1$, $w(1) = 0$ and $w(\rho) \leq v(\rho)$ for any $\rho \in [0,1]$ (see Figure 3.1).

Let us now study the properties of the Filippov solution $y^k(t)$ to (3.2).

PROPOSITION 3.3. *Let $y^k(t)$ be the solution of system (3.2) together with*

$$\|\rho^k(t,\cdot)\|_{L_1} \leq \|\rho_0\|_{L_1},$$
$$0 \leq \rho^k(t,x) \leq 1,$$
$$\|\rho^k(t,\cdot)\|_{\mathcal{BV}} \leq e^{C_0\triangle t}\|\rho_0^k(\cdot)\|_{\mathcal{BV}},$$

for any $t \in [0,\triangle t]$. Then there exists a constant M, independent from k, such that, for any k and for a.e. $t \in [0,\triangle t]$,

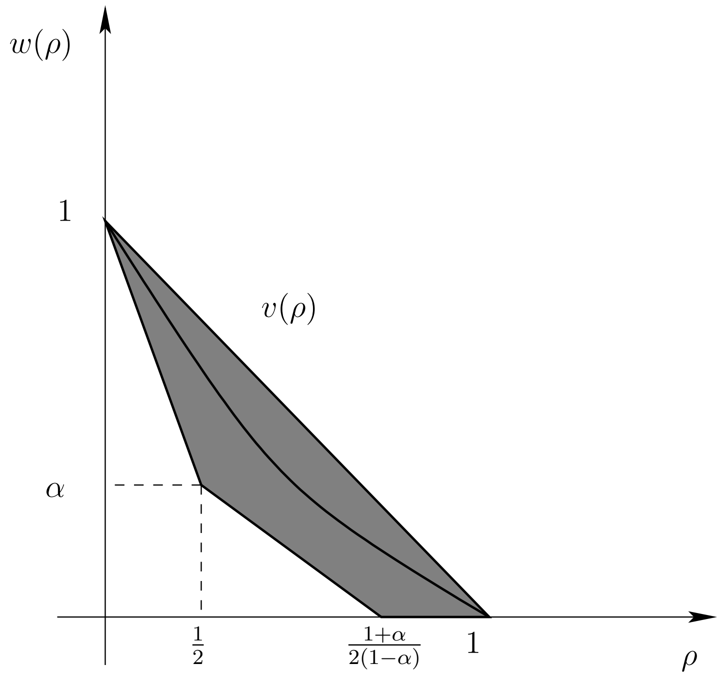

FIG. 3.1. *Any $w(\rho)$ with $w(0) = 1$, $w(1) = 0$ and whose graph is in the grey region is admissible in Example 3.2 ($\alpha < 1/3$)*

(i) $|\dot{y}^k(t)| \leq M$
(ii) $|y^k(t)| \leq |y^{k-1}(\triangle t)| + M\triangle t$.

Proof.

(i) By definition of Filippov solution, we have

$$\dot{y}^k(t) \in \overline{\text{co}}\{w(\rho^k) : \rho^k \in \mathcal{I}[\rho^k(t, y^k(t)-), \rho^k(t, y^k(t)+)]\},$$

where the set $\mathcal{I}[a, b]$ is defined as the smallest interval containing a and b, thus $\mathcal{I}$ is a segment contained in $[0, 1]$ since $0 \leq \rho(t, y) \leq 1$. Since w is a continuous function,

$$\overline{\text{co}}\{w(\rho^k) : \rho^k \in \mathcal{I}[\rho^k(t, y^k(t)-), \rho^k(t, y^k(t)+)]\} \subseteq [\min_{\rho\in[0,1]} w(\rho), \max_{\rho\in[0,1]} w(\rho)]$$

for almost every $t \in [0, \triangle t]$ and for any k and

$$|\dot{y}^k(t)| \leq \max_{\rho\in[0,1]} |w(\rho)| = M \tag{3.14}$$

for almost every $t \in [0, \triangle t]$ and for any k, which proves (i).

(ii) Since the Filippov solution $y^k(t)$ is also absolutely continuous, by integration we get for any k and for almost every $t \in [0, \triangle t]$

$$|y^k(t)| \leq |y^k(0)| + Mt \leq |y^{k-1}(\triangle t)| + M\triangle t,$$

which concludes the proof.
□

3.3. Convergence of fractional step approximation. We are now ready to prove the convergence of the fractional step approximation. For a fixed $T > 0$, we define $\triangle t$ and n such that $[0,T) = \bigcup_{k=1}^{n} \mathcal{J}_k$, with $\mathcal{J}_k = \big[(k-1)\triangle t, k\triangle t\big)$. Thus, for any $t \in [0,T)$, there exists an unique k such that $t \in \mathcal{J}_k$ and therefore we can define the approximating sequences accordingly:

$$\rho_n(t,x) = \rho^k(t',x), \quad y_n(t) = y^k(t') \tag{3.15}$$

when $t' \in [0, \triangle t)$, $k = 1, \ldots, n$.

THEOREM 3.4. *Let ρ_n and y_n be as in (3.15) and assume $\rho_0 \in \mathcal{BV}$, $0 \le \rho_0(x) \le 1$. Then, passing if necessary to subsequences,*

$$\rho_n(t,\cdot) \to \rho(t,\cdot) \ \textit{in } L^1_{\mathrm{loc}}, \ \textit{for a.e. } t \in [0,T]; \tag{3.16}$$

$$(\rho_n)_x \rightharpoonup (\rho)_x \ \textit{in the sense of measures}; \tag{3.17}$$

$$y_n(\cdot) \to y(\cdot) \ \textit{uniformly in } [0,T]; \tag{3.18}$$

$$\dot y_n(\cdot) \rightharpoonup \dot y(\cdot) \ \textit{weakly in } L^1([0,T]). \tag{3.19}$$

Proof. For a.e. $t \in [0,T]$ we get $\rho_n(t,\cdot) = \rho^k(t',\cdot)$, $t' \in (0, \triangle t]$,

$$\begin{aligned}
\|\rho_n(t,\cdot)\|_{\mathcal{BV}} &= \|\rho^k(t',\cdot)\|_{\mathcal{BV}} \\
&\le e^{C_0 \triangle t} \, \|\rho_0^k\|_{\mathcal{BV}} \\
&= e^{C_0\triangle t} \|\rho^{k-1}(\triangle t,\cdot)\|_{\mathcal{BV}} \\
&\le e^{2C_0\triangle t} \|\rho_0^{k-1}(\cdot)\|_{\mathcal{BV}} \\
&\le \cdots \\
&\le e^{C_0 k\triangle t} \|\rho_0(\cdot)\|_{\mathcal{BV}} \\
&\le C(T, y_0, \|\rho_0\|_\infty, \|\rho_0\|_1) \|\rho_0\|_{\mathcal{BV}}.
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
|y_n(t)| &= |y^k(t')| \\
&\le |y^{k-1}(\triangle t)| + M\triangle t \\
&\le |y^{k-1}(0)| + 2M\triangle t \\
&\le \cdots \\
&\le |y_0| + kM\triangle t \\
&\le |y_0| + nM\triangle t \\
&= |y_0| + TM.
\end{aligned}$$

Using Helly's Theorem and a diagonal argument we can get the convergence of $\rho_n(t,\cdot)$ in L^1_{loc} for a countable set of $t \in [0,T]$, up to subsequences. Using the Lipschitz continuity with respect to t for solutions of conservation laws, we can achieve the L^1_{loc} convergence for almost every $t \in [0,T]$.

Furthermore, a direct application of the Ascoli–Arzelà Theorem gives the strong convergences (3.18), up to subsequences. Finally, the uniform estimates (3.20) and (3.14) and the above strong convergences imply (3.17) and (3.19) and the proof is complete. $\square$

3.4. Wave Front Tracking. As it is well known, a solution to the Cauchy problem of a conservation law can be constructed using a Wave Front Tracking algorithm [16]. Roughly speaking, first a $\mathcal{BV}$ initial datum is approximated by a piecewise constant function via sampling (thus with a smaller $\mathcal{BV}$ norm). For small times, a piecewise constant weak solution is obtained by piecing together solutions to Riemann problems, where rarefactions are replaced by a fan of rarefaction shocks. Then, when waves interact, a new Riemann problem is generated and solved again approximating rarefactions by rarefaction shocks' fans and so on.
For details we refer the reader to [4] for the general theory and to [15] for the case of networks.

Our equation $(2.1)_1$ has the flux depending also on t (via $y(t)$) and x, not only ρ, so the Wave Front Tracking algorithm needs to be adapted. (Notice however that in the construction of solutions by fractional step we will only have the dependence on x, since y is kept constant.) We describe below in detail the obtained algorithm for a general equation of the type:

$$\rho_t + g(t, x, \rho)_x = 0,$$

where g is Lipschitz continuous in all variables, under the assumption:

(H) for every (t, x), $g(t, x, \cdot)$ is a concave function of ρ.

Clearly (H) is satisfied by $(2.1)_1$.

We start giving the definition of approximate solution, then show how the adapted Wave Front Tracking algorithm generates such approximate solution.
For a fixed positive integer ν, we call a $\frac{1}{\nu}$-approximate solution (to the Cauchy problem) a function $\bar{\rho}(t, x)$ such that the following conditions hold:

(i) $\bar{\rho}(t, x)$ is piecewise constant, with discontinuities occurring along finitely many Lipschitz curves in the (t, x)-plane. Moreover jumps of $\bar{\rho}(t, x)$ can be shocks or rarefactions and are indexed respectively with $\mathcal{S}(t)$ and $\mathcal{R}(t)$.

(ii) (Velocity Condition) Along each shock $s_\alpha : [t^+, t^-] \to \mathbb{R}$, $\alpha \in \mathcal{S}(t)$, we have

$$\bar{\rho}(t, s_\alpha(t)-) < \bar{\rho}(t, s_\alpha(t)+).$$

Moreover, if $\bar{\rho}_- = \bar{\rho}(t, s_\alpha(t)-)$ and $\bar{\rho}_+ = \bar{\rho}(t, s_\alpha(t)+)$ are the right and left density respectively, then

$$\left|\dot{s}_\alpha(t) - \frac{g(t, s_\alpha(t), \bar{\rho}_+) - g(t, s_\alpha(t), \bar{\rho}_-)}{\bar{\rho}_+ - \bar{\rho}_-}\right| \le \frac{1}{\nu}. \tag{3.21}$$

(iii) Along each rarefaction front $s_\alpha(t)$, $\alpha \in \mathcal{R}(t)$, we have

$$\bar{\rho}(t, s_\alpha(t)+) < \bar{\rho}(t, s_\alpha(t)-) < \bar{\rho}(t, s_\alpha(t)+) + \frac{1}{\nu}.$$

Moreover

$$\dot{s}_\alpha(t) \in [\frac{\partial g}{\partial \rho}(t, s_\alpha(t), \rho(t, s_\alpha(t)-)), \frac{\partial g}{\partial \rho}(t, s_\alpha(t), \rho(t, s_\alpha(t)+))]$$

(iv) For the initial data we have

$$\|\bar{\rho}(0, \cdot) - \rho_0(\cdot)\|_{L^1} < \frac{1}{\nu}.$$

An approximate solution can be constructed as follows by adapted Wave Front Tracking. Given the initial datum ρ_0 in $\mathcal{BV}$, find a sequence $\rho_{0,\nu}$ of piecewise constant functions defined on $\mathbb{R}$ such that $\lim_{\nu\to+\infty}\rho_{0,\nu}=\rho_0$ in $L^1_{loc}([0,1])$ and $TV(\rho_{0,\nu})\le TV(\rho_0)$. This can be done for instance sampling the initial datum on a sequence of close enough points. Then for every $\nu\in\mathbb{N}\setminus\{0\}$, we apply the following procedure. At time $t=0$, we solve all Riemann problems at each discontinuity point of $\rho_{0,\nu}$. We approximate every rarefaction wave with a rarefaction fan, formed by rarefaction shocks (non-entropic shocks) of strength less than $\frac{1}{\nu}$ travelling with the Rankine–Hugoniot speed. All shocks travel along Lipschitz curves. By slightly modifying the speed of waves if necessary, we may assume that, at every positive time t, at most one interaction happens. Then we can repeat the previous construction at every time at which an interaction between waves happens.
Bounds on the number of waves and interactions are directly obtained in the scalar case, in the same way as for the case of flux not depending on t and x. Also $\mathcal{BV}$ bounds are obtained by the same estimates.
The function $\bar\rho(t,x)$ constructed in such way is a $\frac{1}{\nu}$-approximate solution.

Remark 3.5. The estimates and the convergence proved in the previous section refer to *exact* solutions ρ^n to (3.3). However, the same results still hold for Wave Front Tracking approximation of such solutions and therefore from now on we shall refer to approximate Wave Front Tracking solutions in each time step, which will be useful in the sequel.

4. Solution for the coupled model. In this section we prove that the limit functions ρ and y obtained by fractional step method of §3, provide a solution of the PDE-ODE model (2.1) in the sense of Definition 2.2. Since ρ_n and y_n converge strongly to ρ and y, it is straightforward to pass to the limit in the weak formulation of the conservation law and thus we obtain

$$\rho_t+f(x,y(t),\rho)_x=0 \tag{4.1}$$

in the sense of distributions. To study the limit of the ODE in the sense of Filippov, we shall use a stability result for differential inclusions [1]. For completeness, we report here below all notions needed to this purpose.

Definition 4.1. *Let F be a set–valued map with nonempty values (i.e. a proper map) between Hausdorff locally convex spaces X and Y. We say that F is* upper semicontinuous (u.s.c.) at $x^0\in X$ *if for any open N containing $F(x^0)$ there exists a neighborhood M of x^0 such that $F(M)\subset N$. We say that F is* upper semicontinuous *if it is so at every $x^0\in X$.*

Definition 4.2. *Let F be as the definition above and let $\sigma(Y,Y^*)$ be the weak topology of Y. For every $p\in Y^*$, consider the function with values in $]-\infty,+\infty]$ given by*

$$x\to\Sigma(F(x),p)=\sup_{y\in F(x)}\langle p,y\rangle.$$

We say that F is upper hemicontinuous at $x^0\in X$ *if, for every $p\in Y^*$, the function $x\to\Sigma(F(x),p)$ is upper semicontinuous at x^0. We say that F is* upper hemicontinuous *if it is so at every $x^0\in X$.*

Theorem 4.3 ([1]). *Let F be a proper hemicontinuous map from a Hausdorff locally convex space X to the closed convex subsets of a Banach space Y. Let I be an interval of $\mathbb{R}$ and $\xi_n(\cdot)$ and $\eta_n(\cdot)$ be measurable functions from I to X and Y such* *that for almost all $t \in I$ and for every neighborhood $\mathcal{N}$ of 0 in $X \times Y$ there exists $n_0 = n_0(t, \mathcal{N})$ such that for any $n \geq n_0$*

$$(\xi_n(t), \eta_n(t)) \in \text{graph}(F) + \mathcal{N}. \tag{4.2}$$

If

1. *$\xi_n(\cdot)$ converges almost everywhere to a function $\xi(\cdot)$ from I to X,*
2. *$\eta_n(\cdot)$ belongs to $L^1(I, Y)$ and converges weakly to $\eta(\cdot)$ in $L^1(I, Y)$,*

then, for almost all $t \in I$,

$$(\xi(t), \eta(t)) \in \text{graph}(F), \quad \textit{i.e.} \quad \eta(t) \in F(\xi(t)).$$

Now we shall apply the above theorem to the sequences $\xi_n = y_n$ and $\eta_n = \dot{y}_n$ to conclude

$$\dot{y}(t) = w(\rho(t, y(t)))$$

in the sense of Filippov, that is

$$\dot{y} \in \overline{\text{co}}\{w(\rho) : \rho \in \mathcal{I}[\rho(t, y(t)-), \rho(t, y(t)+)]\} \tag{4.3}$$

for almost every $t \in [0, T]$ (see Definition 2.2), redefining ρ to be the set–valued function $(t, x) \mapsto \mathcal{I}[\rho(t, x-), \rho(t, x+)]$ when needed.

For the sake of clarity, we divide the proofs of the hypotheses of Theorem 4.3 in several lemmas.

The first result, taken from [1], states the trivial implication between the notion of semicontinuity with respect to weak topology and hemicontinuity.

Lemma 4.4 ([1]). *Any upper semicontinuous map from X to Y supplied with the weak topology is upper hemicontinuous.*

Lemma 4.5. *The function $(t, x) \mapsto F(t, x) = \overline{\text{co}}\{w(\rho) : \rho \in \mathcal{I}[\rho(t, x-), \rho(t, x+)]\}$ is an upper hemicontinuous set–valued map.*

Proof. Since ρ is a solution to a scalar conservation law, with strictly concave flux (see condition (H) at page 11), we can apply Theorems 11.3.1, 11.3.2, 11.3.3 and 11.3.4 of [12]. In particular we can distinguish three cases: (t, x) is a point of approximate continuity, (t, x) is a point of jump discontinuity or (t, x) is an irregular point. In the first case ρ is continuous at (t, x) by Theorem 11.3.2. In the second case, by Theorem 11.3.3, there exists a Lipschitz curve through (t, x) and ρ is continuous on both connected components in the complement of the curve support. In particular, ρ is upper semicontinuous at (t, x). In the last case, we have to discuss separately the two domains $D_1 = \{(\tau, y) : \tau \geq t\}$ and $D_2 = \{(\tau, y) : \tau < t\}$. On the domain D_1, we are in the same situation as for a jump discontinuity points by Theorem 11.3.1. Finally, from the proof of Theorem 11.3.4, for domain D_2 we know that $\rho(\tau, y) \in [\rho(t, x-), \rho(t, x+)]$ for (τ, y) comprised between the minimal and maximal backward generalized characteristic. Then we can conclude again that ρ is upper semicontinuous at (t, x).

Now, since w is smooth, we obtain that F if upper semicontinuous and, by using Lemma 4.4, we conclude the proof. □

Using once again the regularity of w, it is easy to prove the following result.

Lemma 4.6. *Assume that, for any $\varepsilon > 0$, there exists n_0 such that for any $n \geq n_0$ we have*

$$\text{graph}\{\rho_n(t, \cdot)\} \subseteq \text{graph}\{\rho(t, \cdot)\} + \varepsilon B, \tag{4.4}$$

for almost every t, where B is a unitary ball centered in 0, and ρ_n is the Wave Front Tracking solution of (3.3) *in an appropriate time interval $\mathcal{J}_k$ (see* (3.15)*). Then the condition* (4.2) *holds.*

The proof of (4.4) relies on structural stability properties of (Wave Front Tracking approximations of) weak solutions to conservation laws contained in [5] (see Propositon 4.7 and Proposition 4.8 below). However, our Wave Front Tracking approximation ρ_n refers to the *approximate* equation (3.3), instead of the limit equation (4.1), that is ρ_n is a Wave Front Tracking approximation of

$$\rho_t + f(x, \bar{y}^{k-1}, \rho)_x = \rho_t + \left(\rho v(\rho)\varphi(x - \bar{y}^{k-1})\right)_x = 0,$$

in an appropriate interval $\mathcal{J}_k$, while ρ solves

$$\rho_t + f(x, y(t), \rho)_x = \rho_t + \left(\rho v(\rho)\varphi(x - y(t))\right)_x = 0.$$

Thus, the error on the equation for the density ρ_n is due to the presence of $\bar{y}^{k-1}$ instead of $y(t)$ in the expression of the flux. The difference between the two fluxes is then given by

$$\begin{aligned}\|f(x, \bar{y}^{k-1}, \rho^k) - f(x, y, \rho^k)\|_\infty &= \|\rho^k v(\rho^k)(\varphi(x - \bar{y}^{k-1}) - \varphi(x - y))\|_\infty \\ &\leq \|\rho^k v(\rho^k)\|_\infty \|\varphi'\|_\infty \|\bar{y}^{k-1} - y\|_\infty,\end{aligned}$$

where ρ^k and y^k are defined as in (3.15). In view of the uniform L_∞ bound of ρ^k and the uniform convergence (3.18) for y_n, for any $\varepsilon > 0$ we can choose n large enough to conclude

$$\|f(x, \bar{y}^{k-1}, \rho^k) - f(x, y, \rho^k)\|_\infty \leq C\varepsilon.$$

Moreover, with the same arguments,

$$\begin{aligned}&\|f_\rho(x, \bar{y}^{k-1}, \rho^k) - f_\rho(x, y, \rho^k)\|_\infty = \\ &\qquad = \|v(\rho^k)\varphi(x - \bar{y}^{k-1}) + \rho^k v'(\rho^k)\varphi(x - \bar{y}^{k-1}) + \\ &\qquad\quad - v(\rho^k)\varphi(x - y) - \rho^k v'(\rho^k)\varphi(x - y)\|_\infty \\ &\qquad \leq \|\varphi'\|_\infty(\|v\|_\infty \|\bar{y}^{k-1} - y\|_\infty) + \|\varphi'\|_\infty(\|\rho^k\|_\infty \|v'\|_\infty \|\bar{y}^{k-1} - y\|_\infty) \\ &\qquad \leq \|\varphi'\|_\infty(\|v\|_\infty + \|\rho^k\|_\infty \|v'\|_\infty)\|\bar{y}^{k-1} - y\|_\infty \\ &\qquad \leq C_1\varepsilon\end{aligned}$$

because v is smooth.

Finally, since (uniformly small) errors in speed of propagation of waves are admitted in Wave Front Tracking approximations, condition (3.21) holds also if we replace $\bar{y}^{k-1}$ by y. In other words ρ_n is a Wave Front Tracking approximation also of (4.1).

For completeness and convenience of the reader, we report here below the following results, proved in a more general setting in [5], adapted for our approximation ρ_n to (4.1). Proposition 4.7 concerns stability of shock curves for L^1_{loc} approximations of solutions to conservation laws. The first item has been extracted from [5, Theorem 3], while others both from [5, Theorem 5]. Even if we shall not use all properties of Proposition 4.7, we prefer to list them to underline the "rigidity" of (sequences of) $\mathcal{BV}$ solutions to conservation laws in presence of big shocks and what kind of "shape stability" one can expect for such solutions. The second property, contained in Proposition 4.8, comes from [5, Proposition 13]; it is the convergence of atomic parts of measures $(\rho_n)_x$ and $(\rho)_x$ and relies on uniform $\mathcal{BV}$ estimates for ρ_n.

PROPOSITION 4.7 (Bressan & LeFloch [5]). *Assume $\rho_0 \in \mathcal{BV}$, $0 \leq \rho_0(x) \leq 1$, let ρ_n be the approximate solution of* (4.1) *constructed above and ρ the limiting solution. Then the following holds:*

(i) *For every ε, there exist a finite number of Lipschitz continuous shock curves of strength greater than ε in the limiting solution ρ, admitting left and right limits and satisfying the Rankine–Hugoniot condition at all but countably many times. The oscillation is of order ε outside the shock curves.*

(ii) *Assume that, for $n \geq 1$, $s_n : [t_n^-, t_n^+] \mapsto \mathbb{R}$ is a shock curve of ρ_n of uniformly large strength: $|\sigma_n(t)| > \varepsilon$ for almost every $t \in [t_n^-, t_n^+]$ and for some fixed $\varepsilon > 0$. Assume also $t_n^- \to t^-, t_n^+ \to t^+$, and $s_n(t) \to s(t)$ for every $t \in]t^-, t^+[$. Then $s(\cdot)$ is a shock curve of the limiting solution ρ. That is for all but countably many times, the derivative $\dot{s}$ exists together with distinct right and left limits ρ^+, ρ^- satisfying Rankine–Hugoniot relations.*

(iii) *Viceversa, let $s : [t^-, t^+] \mapsto \mathbb{R}$ be a shock curve of ρ with strength $|\sigma(t)| \geq \varepsilon$ for almost every $t \in [t^-, t^+]$ and for some fixed $\varepsilon > 0$. Then there exists a sequence of shock curves $s_n : I_n \mapsto \mathbb{R}$ of ρ_n such that*

$$\limsup_{n\to\infty} I_n \supseteq [t^-, t^+], \quad |\sigma_n(t)| \geq c_0\varepsilon, \quad \lim_{n\to\infty} s_n(t) = s(t)$$

for some constant $c_0 > 0$ and for almost every $t \in [t^-, t^+]$.

PROPOSITION 4.8 (Bressan & LeFloch [5]). *Let ρ_n be the right continuous version of the approximate solution of* (4.1) *constructed above and assume $\rho_0 \in \mathcal{BV}$, $0 \leq \rho_0(x) \leq 1$. Then, if we truncate the support of ρ_n in a uniformly bounded interval $[-M, M]$, we can conclude the atomic part of the measure obtained as weak limit of $\rho_n(t, \cdot)_x$ coincides with the atomic part of the measure $\rho(t, \cdot)_x$ for all $t \geq 0$, where $\rho(t, \cdot)$ is the right continuous version of the limit of $\rho_n(t, \cdot)$.*

We are now ready to prove the following

LEMMA 4.9. *Let ρ_n be (the right continuous version of) the approximate solution of* (4.1) *constructed above and assume $\rho_0 \in \mathcal{BV}$, $0 \leq \rho_0(x) \leq 1$. Then* (4.4) *holds for almost every time.*

Proof. In what follows, we shall always consider right–continuous representatives of the involved $\mathcal{BV}$ functions. Let us indicate by $\mu_{n,t}^{\pm}$ the positive, resp. negative, part of the measure $\rho_n(t, \cdot)_x$, by $\mu_{\infty,t}^{\pm}$ their weak limits and by $\mu_t^{\pm}$ the positive, resp. negative, part of the measure $\rho(t, \cdot)_x$. Fix $\epsilon > 0$. Since $\mu_{\infty,t}^{\pm}$ and $\mu_t^{\pm}$ are finite Borel measures, there exists M and a finite number of intervals I_h, $h = 1, \ldots, N$, such that the following holds:

($\heartsuit$) $\cup_h I_h \supset [-M, M]$.

($\diamondsuit$) It holds

$$\mu_{\infty,t}^{\pm}(\mathbb{R} \setminus [-M, M]) < \epsilon, \qquad \mu_t^{\pm}(\mathbb{R} \setminus [-M, M]) < \epsilon.$$

Moreover, by weak convergence of $\mu_{n,t}^{\pm}$ to $\mu_{\infty,t}^{\pm}$, for n sufficiently large we get the same estimates for $\mu_{n,t}^{\pm}$.

($\clubsuit$) For every I_h, $h = 1, \ldots, N$, either

$$\mu_{\infty,t}^{+}(I_h) < \epsilon$$

or there exists $\bar{x} \in I_h$ such that

$$\mu^{+}_{\infty,t}(I_h \setminus \{\bar{x}\}) < \epsilon.$$

The same conclusion holds for $\mu^{-}_{\infty,t}$ and $\mu^{\pm}_{t}$. Moreover, if the first case holds true for $\mu^{\pm}_{\infty,t}$, then by weak convergence of $\mu^{\pm}_{n,t}$ to $\mu^{\pm}_{\infty,t}$, for n sufficiently large we get the same estimates $\mu^{\pm}_{n,t}$.

Define $I_{N+1} = (-\infty, -M]$ and $I_{N+2} = [M, \infty)$. Notice that ρ has at most one discontinuity of strength greater than ϵ in each I_h and, apart from that discontinuity, the total variation of ρ is less than ϵ. Thanks to (i) of Proposition 4.7, possibly discarding a countable number of times, each discontinuity in I_h of strength greater than ϵ is a shock satisfying the Rankine–Hugoniot condition. Since the $\mathcal{BV}$ norm of ρ_n is uniformly bounded, then using the stability result (iii) of Proposition 4.7 and Proposition 4.8, the same property is true for the approximating Wave Front Tracking sequence ρ_n, for n sufficiently large. More precisely, if ρ does not have a discontinuity of strength greater than ϵ in I_h, then the same is true for ρ_n and the total variation of ρ_n is less than ϵ in I_h for n sufficiently large.

Thus it is sufficient to construct the neighborhood of the graph of ρ needed to prove (4.4) for $x \in I_h$ with I_h fixed, since the number of such intervals is finite. We distinguish the construction according to the presence of a "big" shock in I_h.

Case 1. Suppose now that ρ does not contain any shock of strength greater than ϵ in I_h, i.e. we have either $h > N$, then ($\diamondsuit$) applies, or we are in the first case of ($\clubsuit$) for $\mu^{\pm}_{t}$. If the latter holds, since from Proposition 4.8 we have that the atomic parts of $\mu^{\pm}_{\infty,t}$ and $\mu^{\pm}_{t}$ coincide, then we are in the first case of ($\clubsuit$) also for the measures $\mu^{\pm}_{\infty,t}$. Therefore, for n sufficiently big, the same conclusion holds also for $\mu^{\pm}_{n,t}$.

Fix a $\bar{x} \in I_h$ such that $\rho_n(t, \bar{x}) \to \rho(t, \bar{x})$, as $n \to +\infty$, then

$$\begin{aligned}
|\rho_n(t,x) - \rho(t,x)| &= |\rho_n(t,x) - \rho_n(t,\bar{x}) + \rho_n(t,\bar{x}) - \rho(t,x) - \rho(t,\bar{x}) + \rho(t,\bar{x})| \\
&\le |\rho_n(t,x) - \rho_n(t,\bar{x})| \\
&\quad + |\rho(t,x) - \rho(t,\bar{x})| + |\rho_n(t,\bar{x}) - \rho(t,\bar{x})| \\
&\le 3\epsilon,
\end{aligned}$$

for n sufficiently large. The above inequality clearly proves (4.4) for any $x \in I_h$ taking $3\epsilon < \varepsilon$.

Case 2. Suppose $I_h = [a_h, b_h]$ contains a shock of strength σ greater than ϵ, then $h \le N$ and let $s(t) \in (a_h, b_h)$ be the position of the shock (here we assume the end points of the interval I_h are points of continuity for $\rho(t, \cdot)$). Thanks to Proposition 4.8, the atomic parts of $\mu^{\pm}_{\infty,t}$ and $\mu^{\pm}_{t}$ coincide and therefore $\mu^{+}_{\infty,t}$ has mass σ at $s(t)$. Moreover, by ($\clubsuit$) it holds:

$$\begin{aligned}
&\mu^{\pm}_{t}(I_h \setminus \{s(t)\}) < \epsilon; \\
&\mu^{\pm}_{\infty,t}(I_h \setminus \{s(t)\}) < \epsilon.
\end{aligned}$$

Then, by testing the weak convergence of $\mu^{+}_{n,t}$ toward $\mu^{+}_{\infty,t}$ with a test function supported in a neighborhood of $s(t)$, we can conclude that for n sufficiently big, μ^{+}_{n} must have at least mass $\sigma - \epsilon$ in the interval $[s(t) - \epsilon, s(t) + \epsilon]$.

Alternatively, we can prove the above claim by using the Wasserstein distance W_1, which metrizes the weak convergence of measures in I_h, being I_h bounded (see for instance [20] for details). Indeed, since $\mu^{\pm}_{n,t}$ weakly converges in the sense of measure

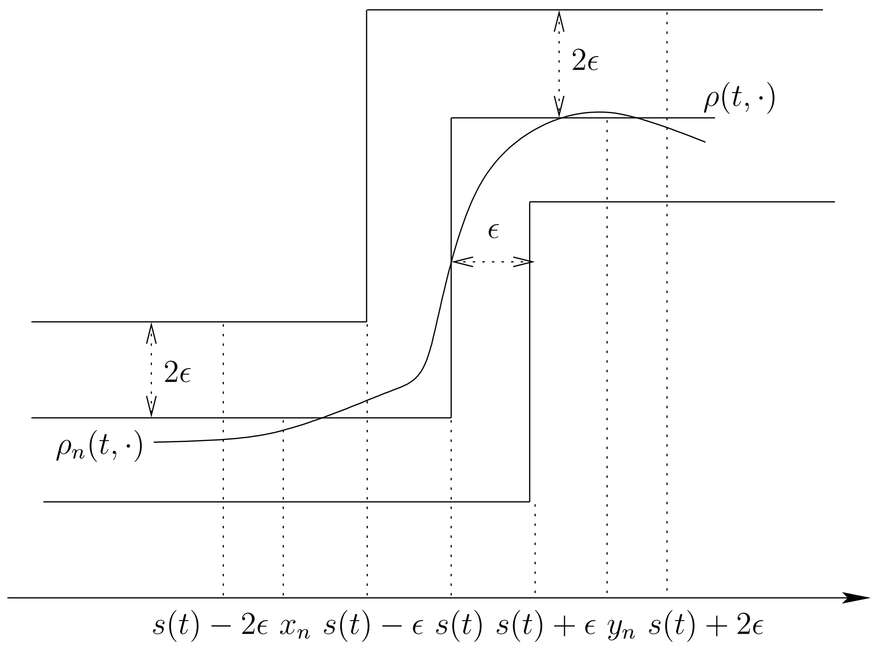

FIG. 4.1. *Construction of the neighborhood for the graph of (the set–valued map) $\rho(t,\cdot)$ close to a shock*

to $\mu^{\pm}_{\infty,t}$, we have:

$$W_1(\mu^{\pm}_{n,t},\mu^{\pm}_{\infty,t}) < \epsilon^3,$$

for n sufficiently big. Therefore, we can conclude the claim is true, because otherwise the (linear) cost defined in W_1 to shift a mass ϵ of $\mu^{+}_{n,t}$ inside the interval $[s(t)-\epsilon, s(t)+\epsilon]$ would be at least of order ϵ^2, i.e.

$$W_1(\mu^{+}_{n,t},\mu^{+}_{\infty,t}) > \epsilon^2,$$

which is a contradiction for ϵ sufficiently small.

Moreover, for n sufficiently big we can find $x_n \in [s(t)-2\epsilon, s(t)-\epsilon[$ and $y_n \in]s(t)+\epsilon, s(t)+2\epsilon]$ such that:

$$|\rho_n(t,x_n)-\rho(t,x_n)| \le \epsilon, \qquad |\rho_n(t,y_n)-\rho(t,y_n)| \le \epsilon.$$

Hence we can conclude that the graph of the approximating sequence ρ_n is in a neighbourhood of size 2ϵ of the graph of ρ, as shown in Figure 4.1. Finally, for the remaining part of the interval I_h we can proceed as in Case 1 and the proof is complete. ▯

Finally, we can prove the main theorem of this section.

THEOREM 4.10. *Assume $\rho_0 \in \mathcal{BV}$, $0 \le \rho_0(x) \le 1$. Then the limit functions ρ and y obtained in Theorem 3.4 are solutions of* (2.1) *and in particular,* $(2.1)_1$ *is verified in the sense of distributions and* $(2.1)_3$ *in the sense of Filippov.*

Proof. As already pointed out before, using the strong convergences (3.16) and (3.18), it is straigthforward to pass to the limit in the conservation law for the density ρ_n and thus obtain $(2.1)_1$ in the sense of distributions, with ρ_0 as initial datum.

Moreover, in view of the convergences (3.18) and (3.19) and in view of Lemma 4.5, Lemma 4.6 and Lemma 4.9, we conclude the hypotheses of Theorem 4.3 are fulfilled for the sequences y_n, $\dot{y}_n$. Therefore we can pass to the limit the differential inclusion

$$\dot{y}_n(t) \in \overline{\text{co}}\{w(\rho^n) : \rho^n \in \mathcal{I}[\rho^n(t, y_n(t)-), \rho^n(t, y_n(t)+)]\}$$

to conclude (4.3) for almost every $t \in [0, T]$. Hence, equation $(2.1)_3$ is verified in the sense of Filippov with y_0 as initial datum and the proof is complete. □

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