Existence of solutions for supply chain models based on partial differential equations

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Abstract

We consider a model for supply chains governed by partial differential equations. The mathematical properties of a continuous model are discussed and existence and uniqueness is proven. Moreover, Lipschitz continuous dependence on the initial data is proven. We make use of the front tracking method to construct approximate solutions. The obtained results extend the preliminary work of [12].

Keywords. Supply chains, networks, front-tracking

AMS Classification. 90B10, 65Mxx

1 Introduction

Supply chain modeling is characterized by different mathematical approaches : on the one hand there are discrete event simulations based on considerations of individual parts; on the other hand, continuous models like [1, 2, 3] using partial differential equations have been introduced. We consider supply chain modeling based on the latter – the continuous models. Recently those models based on scalar conservation laws have been reformulated in the framework of network models where the dynamics on the arcs is governed by a partial differential equation, see [12]. This approach is inspired by other recent discussions on networks, see e.g. [4, 8, 13, 14].

We recall the basic supply chain model under consideration: a supply chain network consists of connected suppliers which are going to process

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parts. Further, each supplier is consists of a processor for assembling and construction and a buffer for unprocessed parts, called queue. We define now

Definition 1.1 (Network definition) *A supply chain network is a finite, connected directed graph consisting of a finite set of arcs $\mathcal{J}$ and a finite set of vertices $\mathcal{V}$. Each supplier j is modelled by an arc j, which is again parameterized by an interval $[a_j, b_j]$.*

Each processor is characterized by a maximum processing capacity μ_j, its length L_j and the processing time T_j. The rate L_j/T_j describes the processing velocity and we assume for simplicity that $L_j/T_j = 1$ for all j. To model the evolution of parts inside the processor we introduce the function $\rho_j(x,t)$, i.e., the density of parts in processor j at point x and time t. Now, the dynamics of each processor on an arc j are governed by an advection equation as in [2]:

$$\partial_t \rho_j(x,t) + \partial_x \min\{\mu_j, \frac{L_j}{T_j}\rho_j(x,t)\} = 0 \quad \forall x \in [a_j, b_j],\ t \in \mathbb{R}^+ \tag{1.1a}$$

$$\rho_j(x,0) = \rho_{j,0}(x) \quad \forall x \in [a_j, b_j]. \tag{1.1b}$$

Equation (1.1) can be derived from a discrete event simulation ([2]) and allows for the following interpretation: The parts are processed with velocity L_j/T_j but with a maximal flux of μ_j. The dynamical behavior of the queues is discussed in detail in the following sections. Roughly speaking, if the inflow is greater than the maximum possible outflow then the queue increases proportionally to the difference of the two, while it decreases in the opposite case.

First, in Section 2, we consider a chain–like network geometry as in Figure 1 for which the discussion below simplifies. Then, in the following section, we turn to the situation of arbitrary networks in particular those with vertices having more than two connected arcs.

Our main achievement is the extension of results proposed in [12]. The correct space to be considered is that of couples (ρ_j, q_j): density of parts and queue buffer occupancy. We prove existence and uniqueness of weak solutions for a general network of supply chains and BV initial data. The densities ρ_j are Lipschitz continuous in time w.r.t. the L^1 metric, while the queues buffer occupancies q_j are absolutely continuous.
Moreover, we prove Lipschitz continuous dependence on the initial data. This, in turn, permits to extend the corresponding semigroup trajectories to L^∞ initial data.

The main idea of the proof is to construct approximated solutions by Front-Tracking [5] and derive bounds on the total variation by a careful estimate of the interactions at the vertices of the network. The proof of Lipschitz dependence uses the approach as in [6].

2 Consecutive processors

In this section we recall the supply chain network model introduced and investigated in [12] and extend the existence results obtained therein.

First, we consider the case where each vertex is connected to exactly to one incoming arc and one outgoing arc and we assume that the arcs are consecutively labeled, i.e., arc j is connected to arc $j+1$, and that $b_j = a_{j-1}$, see also Figure 1.

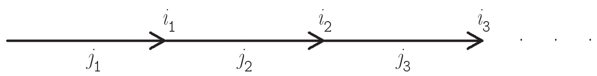

Figure 1: Example of a simple network structure

2.1 Modelling and previous results

As in the introduction, the supplier j is defined by a queue j and a processor j. Physically, the queue is located in front of each processor, i.e., at $x = a_j$. To avoid technical difficulties, we assume that the first supplier consists of a processor only and the last has infinite length, so $a_1 = -\infty$ and $b_N = +\infty$ for the first and, respectively, the last supplier in the supply chain.

In addition to equation (1.1), the queue buffer occupancy in front of each processor is modelled as time-dependent function $t \to q_j(t)$. If the capacity of processor $j-1$ and the demand of processor j are not equal, the queue q_j in- or de-creases its buffer. Mathematically, this implies that each queue q_j satisfies the following equation:

$$\partial_t q_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1}, t)) - f_j(\rho_j(a_j, t)), \quad j = 2, \dots, N. \tag{2.1}$$

Last, a reasonable mathematical condition for the boundary values for outgoing arcs j is given by (see [12]):

$$f_j(\rho_j(a_j, t)) = \begin{cases} \min\{f_{j-1}(\rho_{j-1}(b_{j-1}, t)), \mu_j\} & q_j(t) = 0 \\ \mu_j & q_j(t) > 0 \end{cases} \tag{2.2}$$

This allows for the following interpretation: If the outgoing buffer is empty, we process as many parts as possible but at most μ_j. If the buffering queue contains parts, then we process at the maximal possible rate, namely again μ_j. Finally, the supply chain model is a coupled system of partial and ordinary differential equations on a network given by (1.1, 2.1 and 2.2).

We recall some preliminary facts from [12].
Note that due to the very special flux function

$$f_j(x) := \min\{\mu_j; L_j/T_j\rho\}, \tag{2.3}$$

a Riemann problem for (1.1) and $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ admits one of the following two solutions. Let $\rho_{j,0}(x) = \rho_l$ for $x < 0$ and $\rho_{j,0} = \rho_r$ for $x \geq 0$. If $\rho_l < \rho_r$ then the solution ρ_j is given by

$$\rho_j(x,t) = \begin{cases} \rho_l & -\infty < \frac{x}{t} \leq \frac{f_j(\rho_r)-f_j(\rho_l)}{\rho_r-\rho_l} \\ \rho_r & \frac{f_j(\rho_r)-f_j(\rho_l)}{\rho_r-\rho_l} < \frac{x}{t} < \infty \end{cases} \tag{2.4}$$

If, on the contrary, $\rho_l > \rho_r$, then the following happens. If either $\rho_l \leq \mu_j$ or if $\rho_r \geq \mu_j$ then the solution is given by (2.4). Otherwise (i.e. if $\rho_r < \mu_j < \rho_l$) we obtain the solution given by

$$\rho(x,t) = \begin{cases} \rho_l & -\infty < \frac{x}{t} \leq \frac{f_j(\rho_l)-\mu_j}{\rho_l-\mu_j} \\ \mu_j & \frac{f_j(\rho_l)-\mu_j}{\rho_l-\mu_j} < \frac{x}{t} \leq \frac{\mu_j-f_j(\rho_r)}{\mu_j-\rho_r} \\ \rho_r & \frac{\mu_j-f_j(\rho_r)}{\mu_j-\rho_r} < \frac{x}{t} < \infty \end{cases} \tag{2.5}$$

Notice that the RHS of the first two inequalities is always 0 or 1.
We can introduce the following definition:

Definition 2.1 (Network solution) *A family of functions $\{\rho_j, q_j\}_{j\in\mathcal{J}}$ is called an* admissible *solution for a network as in Figure 1 if, for all j, ρ_j is a weak entropic solutions ([16]) to (1.1), q_j is absolutely continuous and, in the sense of traces for ρ_js, equations (2.1) and (2.2) hold for a.e. t.*

For the particular situation of a single vertex $v \in \mathcal{V}$ with incoming arc $j = 1$ and outgoing arc $j = 2$ and constant initial data $\rho_{j,0}(x) \leq \mu_j$, there exists an admissible solution $\{\rho_1, \rho_2, q_2\}$. The solution has the following explicit form

$$\rho_1(x,t) = \rho_{1,0} \tag{2.6a}$$

$$\rho_2(x,t) = \begin{cases} f_1(\rho_{1,0}) < \mu_2 & \begin{cases} \rho_{1,0} & 0 \le (x-t_0)/t < 1 = \frac{f_2(\mu_2)-f_2(\rho_{1,0})}{\mu_2-\rho_{1,0}} \\ \mu_2 & 1 \le (x-t_0)/t \text{ and } x/t < 1 \\ \rho_{2,0} & 1 \le x/t < \infty \end{cases} \\ f_1(\rho_{1,0}) \ge \mu_2 & \begin{cases} \mu_2 & 0 \le x/t < 1 = \frac{f_2(\mu_2)-f_2(\rho_{2,0})}{\mu_2-\rho_{2,0}} \\ \rho_{2,0} & 1 \le x/t < \infty \end{cases} \end{cases} \tag{2.6b}$$

$$q_2(t) = q_{2,0} + \int_0^t f_1(\rho_{1,0}) - f_2(\rho_2(a_2+,\tau))d\tau \tag{2.6c}$$

wherein $t_0 = q_{2,0}/\left(\mu_2 - f_1(\rho_{1,0})\right)$. For a network as in Figure 1, for initial data $\{\rho_{j,0}(x)\}_j$ where each $\rho_{j,0}$ is a step function, and for initial values $q_j(0) = 0$, there exists an admissible solution $\{\rho_j, q_j\}_j$ to the network problem (1.1,2.1, 2.2), see [12]. The construction of the solution is based on wave- or front–tracking (see below and in [9, 5, 15]. For applications of this method in context with network problems we also refer to [14, 8].

2.2 Wave front tracking approximations

To start, we introduce a equi–distant grid $(i\delta)_{i0}^{N_x}$ such that $0 \le (i\delta) \le \max\{\mu_j : j \in \mathcal{J}\}$ and such that $\forall j \exists i_j : i\delta\mu_j$. Here, it is implicitely assumed that μ_i/μ_j is rational. We approximate the initial data by step functions $\rho^\delta_{j,0}$ taking values in the set $\{i\delta : i0, \dots, N_x\}$. Then each Riemann problem inside an arc or at a vertex is solved, obtaining various traveling discontinuities. If discontinuities collide, then the collision can be resolved by either solving a Riemann problem inside the arc j (see equations (2.4),(2.5)) or as a collision with a vertex (see equations (2.6)). In both cases we obtain new discontinuities propagating until the next collision.
At the same time, an evolution of the queues buffers q_j are automatically defined when solving the Riemann problems at vertices.
This construction guarantees that the solution on arcs takes values only in the set $\{i\delta : i = 0, \dots, N_x\}$ and we obtain a wave front tracking approximate solution (denoted by $(\rho^\delta, q^\delta) := \{(\rho^\delta_j, q^\delta_j)\}_j$) consisting of a set of moving discontinuities along the intervals $[a_j, b_j]$ and queues buffers evolutions.

As usual ([5],[10]), to guarantee the good definition of wave front tracking approximate solutions and, passing to the limit, prove existence of solutions in the sense of Definition 2.1, three basic estimates are in order:

1. Estimate on the number of waves;

2. Estimate on the number of interactions (between waves and of waves with queues);
3. Estimate on total variation of solutions for ρ_j;

Moreover, in our case, we need to prove some compactness of the sequence q_j^δ in an appropriate space.
It is easy to check that every collision inside an arc decreases the number of waves, while the interactions with a vertex may produce two new waves, c.f. equation (2.6). Also, the characteristic velocity of waves is always positive and is bounded from above, then the first two estimates are readily obtained, see [12]. Therefore the construction of wave front tracking approximations is well–defined up to any given time T.

2.3 Total variation estimates on densities

Here, we provide total variation estimates on ρ_j^δ (i.e. along wave-front tracking approximate solutions.) This will imply the existence of an admissible solution for BV-initial data $\rho_{j,0}$.

First, we discuss the case of initial data $\rho_{j,0}$ additionally satisfying next assumption (K):

(K) For every j the initial datum satisfies $\rho_{j,0} \leq \mu_j$.

The above construction guarantees that (K) remains valid for every time along wave front tracking approximate solutions.

Each $\rho_j^\delta(x,t)$ is a piecewise constant function in x and thus will define a number of constant states $\rho_{j,i}^\delta, i = 1, \dots, N_j$, where we assume that $\rho_j^\delta(a_j, \cdot) = \rho_{j,1}^\delta$ and so forth. We define the total variation of the flux on the network as

$$T.V.(f(\rho^\delta)) = \sum_{j \in \mathcal{J}} T.V.(f_j(\rho_j^\delta(\cdot,t))) = \sum_{j \in \mathcal{J}} \sum_{i1}^{N_j - 1} |f_j(\rho_{j,i}^\delta) - f_j(\rho_{j,i+1}^\delta)| \quad (2.7)$$

Note that, thanks to assumption (K), a bound on $T.V.(f_j(\rho_j^\delta(\cdot,t)))$ provides also a bound on $T.V.(\rho_j^\delta(\cdot,t))$, since $\rho_{j,i}^\delta \leq \mu_j$ for all j,i. Furthermore, $T.V.(f(\rho^\delta))$ does not increase, when discontinuities collide **inside** an arc j, see [5]. Next, we discuss the collision of a discontinuity with a vertex.

Lemma 2.2 *Assume a single vertex with incoming arc $j = 1$ and outgoing arc $j = 2$. Furthermore, assume constant states $\rho_{j,0}, j = 1, 2$ at the vertex* *and consider a discontinuity colliding at time t_0. Denote the new solution at the vertex after the collision by $\bar{\rho}_j$. Assume no more collision of discontinuities happens until t^*. Then, for all $t_0 < t < t^*$,*

$$\sum_{j=1}^{2} T.V.(f_j(\rho_j(\cdot,t))) + |\partial_t q_2(t)| \leq \sum_{j=1}^{2} T.V.(f_j(\rho_j(\cdot,t_0))) + |\partial_t q_2(t_0)| \quad (2.8)$$

Proof. By construction the colliding discontinuity has to arrive from arc $j = 1$ and therefore the total variation of the flux on this arc decreases by $|f_1(\bar{\rho}_1) - f_1(\rho_{1,0})|$. On the outgoing arc $j = 2$ we distinguish two cases. First, assume that $f_2(\rho_{2,0}) = f_1(\rho_{1,0})$. Then, due to (2.1) we have $\partial_t q_2(t_0) = 0$. If $f_1(\bar{\rho}_1) \leq \mu_2$, then $f_2(\bar{\rho}_2) = f_1(\bar{\rho}_1)$ and (2.8) holds. If on the other hand, $f_1(\bar{\rho}_1) > \mu_2$, then due to (2.2), $f_2(\bar{\rho}_2) = \mu_2$ and again (2.8) holds, since for $t > t_0$:

$$\begin{aligned} |f_1(\bar{\rho}_1) - f_1(\rho_{1,0})| &= |\mu_2 - f_1(\rho_{1,0})| + |f_1(\bar{\rho}_1) - \mu_2| \\ &= |f_2(\bar{\rho}_2) - f_2(\rho_{2,0})| + |\partial_t q_2(t)|. \end{aligned}$$

In the second case, we assume $f_2(\rho_{2,0}) = \mu_2$. Then,

$$|\partial_t q_2(t_0)| = |f_1(\rho_{1,0}) - \mu_2|$$

and we distinguish two more subcases depending whether the queue is increasing or decreasing after the collisions. First, assume $f_1(\bar{\rho}_1) \geq \mu_2$, i.e., the queue q_2 is increasing with

$$|\partial_t q_2(t)| = f_1(\bar{\rho}_1) - \mu_2$$

and

$$f_2(\bar{\rho}_2) = f_2(\rho_{2,0}) = \mu_2. \quad (2.9)$$

Still (2.8) holds, since

$$|f_1(\rho_{1,0}) - f_1(\bar{\rho}_1)| + |\partial_t q_2(t_0)| \geq |\partial_t q_2(t)|$$

for $t > t_0$. Second, assume $f_1(\bar{\rho}_1) < \mu_2$, i.e., the queue q_2 is decreasing. Let $\bar{t}$ be such that $q_2(\bar{t}) = 0$. Then (2.8) holds since for $t < \min\{\bar{t}, t^*\}$:

$$T.V.(f_2(\bar{\rho}_2(\cdot,t))) = 0$$

and

$$\begin{aligned} |f_1(\bar{\rho}_1) - f_1(\rho_{1,0})| + |f_1(\rho_{1,0}) - \mu_2| &\geq |\mu_2 - f_1(\bar{\rho}_1)| \\ &= |\partial_t q_2(t)|. \end{aligned}$$

If $\bar{t} < t^*$ we obtain a new travelling discontinuity on the outgoing arc $j = 2$ for times $t > \bar{t}$ when the queue q_2 becomes empty: $f_2(\bar{\rho}_2(a_2+, t)) = f_1(\bar{\rho}_1)$ and $\partial_t q(t) = 0$ for $t > \bar{t}$. Then, (2.8) still holds, since

$$T.V.(f_2(\rho_2(\cdot, t))) + |\partial_t q_2(t)|$$

is constant for this interaction. This finishes the proof. □

Summarizing, we conclude that for all $\delta > 0$ the following holds for all $t > 0$:

$$\sum_{j=1}^{N} T.V.(\rho_j^\delta(\cdot, t)) + \sum_{j=2}^{N} |\partial_t q_j^\delta(t)| \leq \sum_{j=1}^{N} T.V.(\rho_{j,0}^\delta(\cdot)) + \sum_{j=2}^{N} |\partial_t q_j^\delta(0)| \quad (2.10a)$$

$$\text{and } \rho_j^\delta(x, t) \leq \max_j \mu_j \ \forall j, x. \quad (2.10b)$$

2.4 Total variation estimates on queues buffers

Let us now pass to estimate the total variation of $\partial_t q_j$:

Lemma 2.3 *Assume a single vertex with incoming arc $j = 1$ and outgoing arc $j = 2$ (of infinite length). Furthermore, assume constant states $\rho_{j,0}, j = 1, 2$ at the vertex and consider a discontinuity colliding at time t_0. Denote the new solution at the vertex after the collision by $\bar{\rho}_j$. Assume no more collision of discontinuities happens until t^*. Then, for all $t_0 < t < t^*$,*

$$T.V.(\partial_t q_2, [t_0, t]) \leq 2\ |f_1(\bar{\rho}_1) - f_1(\rho_{1,0})| + |\partial_t q_2(t_0)|. \quad (2.11)$$

Proof. The interactions are clearly the same examined in Lemma 2.2.
First, assume that $f_2(\rho_{2,0}) = f_1(\rho_{1,0})$. Then, due to (2.1) we have $\partial_t q_2(t_0) = 0$. If $f_1(\bar{\rho}_1) \leq \mu_2$, then $f_2(\bar{\rho}_2) = f_1(\bar{\rho}_1)$ and $\partial_t q_2(t) = 0$, thus (2.11) holds because the left hand side vanishes.
If on the other hand, $f_1(\bar{\rho}_1) > \mu_2$, then:

$$|f_1(\bar{\rho}_1) - f_1(\rho_{1,0})| = |f_2(\bar{\rho}_2) - f_2(\rho_{2,0})| + |\partial_t q_2(t)| \geq |\partial_t q_2(t)|,$$

thus (2.11) holds because $\partial_t q_2(t_0) = 0$.
In the second case, we assume $f_2(\rho_{2,0}) = \mu_2$. Then,

$$\partial_t q_2(t_0) = f_1(\rho_{1,0}) - \mu_2, \quad \partial_t q_2(t_0+) = f_1(\bar{\rho}_1) - \mu_2. \quad (2.12)$$

If the queue is increasing after the interaction, then:

$$T.V.(\partial_t q_2(t), [t_0, t]) = |f_1(\rho_{1,0}) - \mu_2 - (f_1(\bar{\rho}_1) - \mu_2)| = |f_1(\rho_{1,0}) - f_1(\bar{\rho}_1)|. \quad (2.13)$$

Second, assume $f_1(\bar\rho_1) < \mu_2$, i.e., the queue q_2 is decreasing. Let $\bar t$ be such that $q_2(\bar t) = 0$. For $t < \min\{\bar t, t^*\}$, (2.13) still holds, thus we conclude in the case $t^* \le \bar t$. If, on the contrary, $\bar t < t^*$ we obtain a new travelling discontinuity on the outgoing arc $j = 2$ for times $t > \bar t$ when the queue q_2 becomes empty: $f_2(\bar\rho_2(a_2+, t)) = f_1(\bar\rho_1)$ and $\partial_t q_2(t) = 0$ for $t > \bar t$. Then,

$$T.V.(\partial_t q_2(t), [t_0, t]) = |\partial_t q_2(t_0) - \partial_t q_2(t_0+)| + |\partial_t q_2(t_0+) - \partial_t q_2(t)|$$

$$\le |f_1(\rho_{1,0}) - f_1(\bar\rho_1)| + |f_1(\bar\rho_1) - \mu_2| \le$$

$$2\ |f_1(\rho_{1,0}) - f_1(\bar\rho_1)| + |\partial_t q_2(t_0)|.$$

□

We can now reason as follows. Define:

$$\eta = \min_j |b_j - a_j|$$

the minimum length of a supplier and set

$$TV_j^k = T.V.(f_j(\rho_j^\delta(\cdot, k\eta))), \qquad q_j^k = \partial_t q_j^\delta(k\eta).$$

Then by Lemmas 2.2 and 2.3, we get:

$$T.V.(\partial_t q_j^\delta, [k\eta, (k+1)\eta]) \le 2\, TV_{j-1}^k + q_j^k,$$

$$q_j^k + TV_{j-1}^k + TV_j^k \le q_j^{k-1} + TV_{j-1}^{k-1} + TV_j^{k-1}.$$

Moreover, defining by $\widetilde{TV}_N^k$ the variation in the flux produced on the last supplier by the queue q_N^δ on the time interval $[k\eta, (k+1)\eta]$, we get:

$$TV_1^k \le TV_1^0, \quad q_N^k + TV_{N-1}^k + \widetilde{TV}_N^k \le q_N^{k-1} + TV_{N-1}^{k-1} + \widetilde{TV}_N^{k-1}.$$

Therefore, summing up on j and k we get the following:

$$\sum_{j=2}^N T.V.(\partial_t q_j^\delta, [0, K\eta]) = \sum_{j=2}^N \sum_{k=0}^{K-1} T.V.(\partial_t q_j^\delta, [k\eta, (k+1)\eta]) \le$$

$$\sum_{j=2}^N \sum_{k=0}^{K-1} \left(2\, TV_{j-1}^k + q_j^k\right) \le K \sum_{j=2}^N \left(2\, TV_{j-1}^0 + q_j^0\right).$$

Restating:

$$\sum_{j=2}^N T.V.(\partial_t q_j^\delta, [0, K\eta]) \le K \sum_{j=2}^N \left(2\ T.V.(\rho_{j-1,0}^\delta(\cdot)) + |\partial_t q_j^\delta(0)|\right)\ \forall t. \quad (2.14)$$

2.5 Existence of a network solution for BV initial data

For existence of solutions, we consider the space the space of data (ρ, q) on the supply chain with the norm

$$\|(\rho, q)\| = \sum_j \|\rho_j\|_{L^1} + \sum_j |q_j|. \tag{2.15}$$

Then, we want to find a solution in the space $Lip([0,T], L^1((a_j, b_j)))$ for the ρ components and in the space $W^{1,1}([0,T])$ for the q components.

Due to the the special flux function, we obtain discontinuities travelling with speed v at most equal to 1. Therefore, we have for $t_1 < t_2$ and every j

$$\int_{a_j}^{b_j} |\rho_j^\delta(x,t_1) - \rho_j^\delta(x,t_2)|dx \leq T.V.(\rho_j^\delta(\cdot,t_1))|t_1 - t_2| + \int_{t_1}^{t_2} |f(\rho_j^\delta(a_j,t))|\, dt. \tag{2.16}$$

The estimate (2.16) guarantees Lipschitz dependence w.r.t. time in L^1, while (2.10) ensures uniform BV bounds. Therefore, by using standard techniques [5, 17], one can show, that for $\delta \to 0$ a subsequence of ρ^δ converges in L^1 provided that $T.V.(\rho_{j,0}(x))$ is bounded. Furthermore, the limit solution ρ^* is a weak entropic solution for (1.1).

For what concerns q_j, we observe that $\partial_t q_j$ are of bounded variation. Again by BV compactness, we have that $\partial_t q_j$ converges by subsequences in BV, in particular almost everywhere and strongly in L^1. Thus q_j converges uniformly. Finally q_j converges by subsequences in $W^{1,1}$.

Remark 2.4 *Notice that we can pass to the limit using the uniform Lipschitz continuities of q_j. In fact, by definition, $Lip_t(q_j) \leq \max\{\mu_{j-1}, \mu_j\}$. Thus we can pass to the limit obtaining Lipschitz continuous functions with the same bound on the Lipschitz constant.*
Also, we can pass to the limit using estimate (2.10) and Ascoli-Arzelá Theorem, but in that case we can not guarantee that $\partial_t q_j$ is in BV and that q_j is in $W^{1,1}$.

Consider now the case in which (K) is violated. For every j, the data entering the supplier from a_j satisfies (K). Consider the generalized characteristic $\pi_j(t)$ starting from a_j at time 0 and let τ_j (possibly $+\infty$) the time in which it reaches b_j. We can divide the supplier in two regions:

$$A_j = \{(t,x) : x \leq \pi_j(t)\}, \qquad B_j = \{(t,x) : x > \pi_j(t)\},$$

see Figure 2.

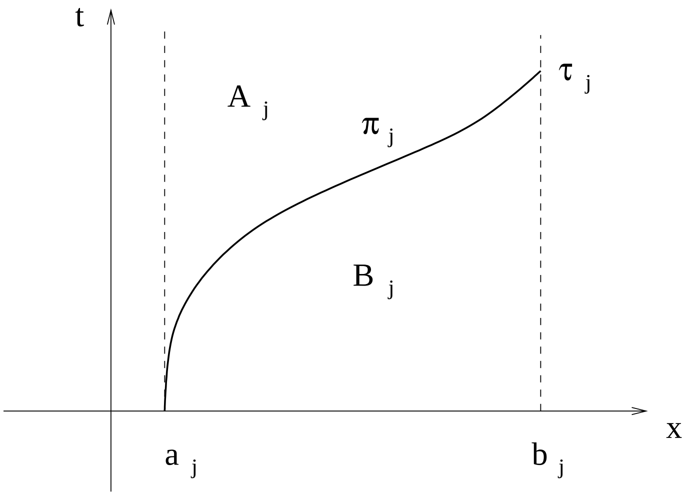

Figure 2: Regions A_j and B_j.

A_j is the region influenced by the incoming flux from a_j, while B_j is the region where ρ depends only on the initial datum $\rho_{j,0}$. Notice that for $t \geq \tau_j$, $B_j \cap \{(t,x) : a_j \leq x \leq b_j\} = \emptyset$. On A_j, (K) holds true thus also the estimate (2.10) holds. While on B_j the solution is the same as the solution to a scalar problem, thus the total variation is decreasing. We thus reach again compactness in BV and the existence of a solution.

Finally, we thus get the following:

Proposition 2.5 *If $T.V.(\rho_{j,0}(x)) \leq C$ for some $C > 0$, then there exists a solution (ρ, q) on the network, such that $(\rho, q) \in Lip([0,T], L^1((a_j, b_j))) \times W^{1,1}([0,T])$, ρ is BV for every time and $\partial_t q_j$ is in BV.*

2.6 Uniqueness and Lipschitz continuous dependence.

We want to prove uniqueness and Lipschitz continuous dependence on the space of data (ρ, q) on the supply chain with the norm (2.15). We use the same approach of [6, 10], thus consider a Riemannian metric on this space, where the tangent vectors are considered only for ρ_j piecewise constant functions.

Let us first focus on the ρ_js: a "generalized tangent vector" consists of two components (v, ξ), where $v \in L^1$ describes the L^1 infinitesimal displacement, while $\xi \in \mathbb{R}^n$ describes the infinitesimal displacement of discontinuities. A family of piecewise constant functions $\theta \to \rho^\theta$, $\theta \in [0,1]$, with the same number of jumps say at the points $x_1^\theta < ... < x_M^\theta$, admits a tangent vector is the following functions are well defined (see Figure 3)

$$L^1 \ni v^\theta(x) \doteq \lim_{h\to 0} \frac{\rho^{\theta+h}(x) - \rho^\theta(x)}{h},$$

and also the numbers

$$\xi_\beta^\theta \doteq \lim_{h\to 0} \frac{x_\beta^{\theta+h} - x_\beta^\theta}{h}, \qquad \beta = 1, ..., M.$$

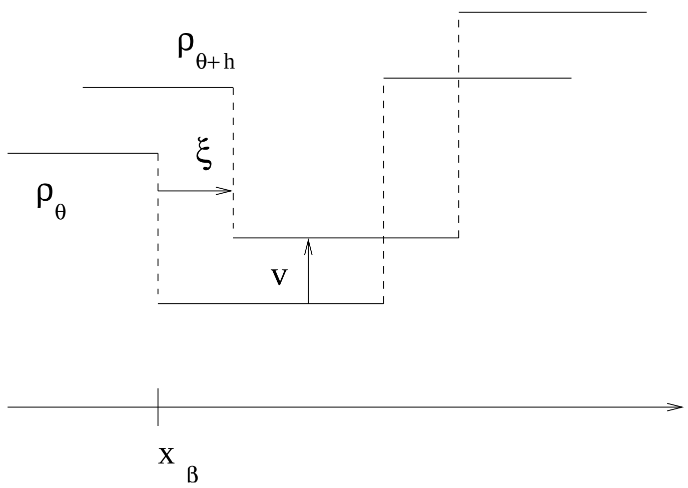

Figure 3: Construction of "generalized tangent vectors".

Notice that the path $\theta \to \rho^\theta$ is not differentiable w.r.t. the usual differential structure of L^1, in fact if $\xi_\beta^\theta \neq 0$, as $h \to 0$ the ratio $\left[\rho^{\theta+h}(x) - \rho^\theta(x)\right]/h$ does not converge to any limit in L^1.
The L^1-length of the path $\gamma : \theta \to \rho^\theta$ in given by:

$$\|\gamma\|_{L^1} = \int_0^1 \left\|v^\theta\right\|_{L^1} d\theta + \sum_{\beta=1}^{M} \int_0^1 \left|\rho^\theta(x_\beta+) - \rho^\theta(x_\beta-)\right| \left|\xi_\beta^\theta\right| d\theta. \tag{2.17}$$

According to (2.17), the L^1-length of a path γ is the integral of the norm of its tangent vector, defined as follows:

$$\|(v,\xi)\| \doteq \|v\|_{L^1} + \sum_{\beta=1}^{M} |\Delta\rho_\beta| \, |\xi_\beta| \, ,$$

where $\Delta\rho_\beta = \rho(x_\beta+) - \rho(x_\beta-)$ is the jump across the discontinuity x_β.

Now, given two piecewise constant functions ρ and ρ', call $\Omega(u, u')$ the family of all "differentiable" paths $\gamma : [0,1] \to \gamma(t)$ with $\gamma(0) = u$, $\gamma(1) = u'$. The Riemannian distance between u and u' is given by

$$d(u, u') \doteq \inf \left\{ \|\gamma\|_{L^1}, \gamma \in \Omega(u, u') \right\}.$$

To define d on all L^1, for given $u, u' \in L^1$ we set

$$\begin{aligned} d(u, u') \doteq \inf \Big\{ & \|\gamma\|_{L^1} + \|u - \tilde{u}\|_{L^1} + \left\|u' - \tilde{u}'\right\|_{L^1} : \\ & \tilde{u}, \tilde{u}' \text{ piecewise constant functions}, \gamma \in \Omega(u, u') \Big\}. \end{aligned}$$

It is easy to check that this distance coincides with the distance of L^1.

To estimate the L^1 distance among wave front tracking approximate solutions we proceed as follows. Take ρ, ρ' piecewise constant initial data and let $\gamma_0(\vartheta) = u^\vartheta$ be a regular path joining $\rho = \rho^0$ with $\rho' = \rho^1$. Define $\rho^\vartheta(t, x)$ to be a wave-front tracking approximate solution with initial data ρ^ϑ and let $\gamma_t(\vartheta) = \rho^\vartheta(t, \cdot)$. Then for every $t \geq 0$, γ_t is a differentiable path. If we can prove that

$$\|\gamma_t\|_{L^1} \leq \|\gamma_0\|_{L^1}, \tag{2.18}$$

for every $t \geq 0$ then

$$\left\|\rho(t,\cdot) - \rho'(t,\cdot)\right\|_{L^1} \leq \inf_{\gamma_t} \|\gamma_t\|_{L^1} \leq \inf_{\gamma_0} \|\gamma_0\|_{L^1} = \left\|\rho(0,\cdot) - \rho'(0,\cdot)\right\|_{L^1}. \tag{2.19}$$

Now, to obtain (2.18), hence (2.19), it is enough to prove that, for every tangent vector $(v, \xi)(t)$ to any regular path γ_t, one has:

$$\|(v, \xi)(t)\| \leq \|(v, \xi)(0)\|, \tag{2.20}$$

i.e the norm of a tangent vector does not increase in time. Moreover, if (2.19) is established, then uniqueness and Lipschitz continuous dependence of solutions to Cauchy problems is straightforwardly achieved passing to the limit on the wave-front tracking approximate solutions.

Remark 2.6 *Since the Riemannian distance d is equivalent to the L^1 metric, the reader could think that the whole framework is not so useful. On the contrary, the different differential structure permits to rely on tangent vectors, whose norm can be easily controlled. This would not be possible using the tangent vectors of the usual differential structure of L^1, i.e. having only the v component.*
Also, while for systems of conservation laws it is possible to find a decreasing functional (see [7]), this is not the case for networks (see [10]), even for a scalar conservation law.

Let us now turn to the supply-chains case. It is easy to see that all paths in L^1 connecting piecewise constant functions can be realized using only the ξ component of the tangent vector, see [5, 6]. Therefore, indicating by $x_{\beta_i^j}$ the positions of discontinuities, $j = 1, \ldots, N$, $i = 1, \ldots, M_j$, a tangent vector to a function defined on the network is given by:

$$(\xi_{\beta_i^j}, \eta_j),$$

where $\xi_{\beta_i^j}$ is the shift of the discontinuity $x_{\beta_i^j}$, while η_j is the shift of the queue buffer occupancy q_j. The norm of a tangent vector is given by:

$$\|(\xi_{\beta_i^j}, \eta_j)\| = \sum_{j,i} |\xi_{\beta_i^j}||\Delta\rho_{\beta_i^j}| + \sum_j |\eta_j|.$$

Again, to control the distance among solutions it is enough to control the evolution of norms of tangent vectors. Finally, we have:

Lemma 2.7 *The norm of tangent vectors are decreasing along wave front tracking approximations.*

Proof. The norm of tangent vectors changes only at interaction times or if a wave is generated, see [6], thus we have to consider three cases:

i) Two waves interact on a supplier.

ii) A wave interacts with a vertex.

iii) One queue empties down.

Case i) is as the classical case, see [5, 6].

Consider case ii) and assume that the interaction happens with vertex j at time t. Let us indicate by $f_j^\pm$ the value of the flux at a_j before and after the interaction and, similarly, by $f_{j-1}^\pm$ the value of the flux at b_{j-1}. In general we use the letters $+$ and $-$ to indicate quantities before and after the interaction, respectively.
Assume first $q_j(t) = 0$, then $f_{j-1}^- = f_j^- < \mu_j$. If $f_{j-1}^+ \leq \mu_j$ then the queue remains empty, a ρ wave is generated on supplier j and the tangent vector norm remains unchanged. If $f_{j-1}^+ > \mu_j$, then $\xi^+ = \xi^-$, $\Delta\rho^+ = \mu_j - f_j^-$ and $\eta^+ = \eta^- + \xi^-(f_{j-1}^+ - \mu_j)$. Since $\Delta\rho^- = f_{j-1}^+ - f_{j-1}^- = f_{j-1}^+ - f_j^-$, the norm is conserved.
Assume now $q_j(t) > 0$, then $f_j^- = f_j^+ = \mu_j$. No ρ wave is produced and $\eta^+ = \eta^- + \xi^-\Delta\rho^-$ and again we conclude.

Let us pass to case iii) and use the same notation of case ii). Then $f_j^- = \mu_j$ and $f_j^+ = f_{j-1}^- = f_{j-1}^+ < \mu_j$. We get $\Delta\rho^+ = \mu_j - f_{j-1}^-$, $\xi^+ = \eta^-/(\mu_j - f_{j-1}^-)$ and $\eta^+ = 0$, thus we are finished. □

2.7 Existence for L^1 initial data

Since we proved Lipschitz continuous dependence, by an approximation argument, we also get existence for L^1 initial data. More precisely, we get:

Theorem 2.8 *There exists a Lipschitz continuous semigroup S_t defined on the domain $\mathcal{D} = \{(\rho_j, q_j) : \rho_j \in L^\infty, q_j \in \mathbb{R}\}$. Moreover, for every initial datum (ρ_j, q_j) with ρ_j of bounded variation, the semigroup trajectory $t \mapsto S_t(\rho_j, q_j)$ is a network solution.*

We point out that assumption (K) guarantees the existence of a solution on the network, while this is not granted in the general case as showed by next example.

Example 2.9 *Consider a simple network formed by only one vertex connecting an incoming arc $j = 1$ and an outgoing arc $j = 2$ and initial data:*

$$\rho_1(0, x) = \mu_1 = \mu_2, \qquad \rho_2(0, x) = \mu_2 + \sin^2\left(\frac{1}{x - a_2}\right), \qquad q_2(0) > 0.$$

Clearly on the outgoing arc $j = 2$ the solution takes values in the flat part of the flux, thus it is constant in time. In particular $\rho_2(t, x)$ has not trace as $x \to a_2$ for any value of t.

Remark 2.10 *Notice that (2.2) still makes sense for Example 2.9 if we interpret the relation to hold for every limit $\lim_n \rho_2(t, x_n)$ with $x_n \to a_2$. On the other side, we can make oscillations in ρ_2 arbitrarily large if we put no constraints on the possible values of ρ_2.*

3 General networks

Now, we turn to the case of more general networks as for example depicted in Figure 4.

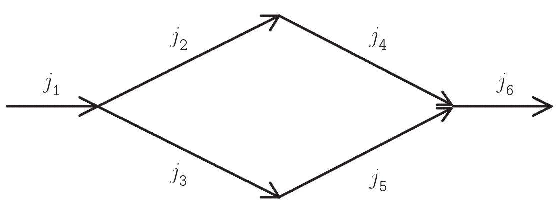

Figure 4: Network geometry for a supply chain

3.1 Modeling

We consider general vertices $v \in \mathcal{V}$ with m_v incoming and n_v outgoing arcs. The set of arc indices of incoming (outgoing) arcs is denoted by δ_v^- (δ_v^+). If we have more than one outgoing arc, we need to define the distribution of the goods from the incoming arcs. Similarly to [8], we model this as follows. We assume that for each single vertex v a matrix $A_v := A(\alpha_{ij})_{i,j} \in \mathbb{R}^{m_v \times n_v}$ is given and that the total flux willing to go to arc $j \in \delta_v^+$ is given by

$$\sum_{i \in \delta_v^-} \alpha_{ij} f_i(\rho_i(b_i-, t)).$$

Therefore, we assume that the matrix A satisfies for all $i \in \delta_v^-, j \in \delta_v^+$: $0 \leq \alpha_{ij} \leq 1$ and $\sum_{j \in \delta_v^+} \alpha_{ij} = 1$. Then, the supply chain network model is given by (1.1) and for each junction v by the following equations for the queues, see also [11],

$$\forall j \in \delta_v^+ : \partial_t q_j(t) = \sum_{i \in \delta_v^-} \alpha_{ij} f_i(\rho_i(b_i-, t)) - f_j(\rho_j(a_j+, t)), \tag{3.1}$$

and the boundary values $\forall j \in \delta_v^+$,

$$f_j(\rho_j(a_j+, t)) = \begin{cases} \min\{\sum_{i \in \delta_v^-} \alpha_{ij} f_i(\rho_i(b_i-, t)); \mu_j\} & q_j(t) = 0 \\ \mu_j & q_j(t) > 0 \end{cases}. \tag{3.2}$$

Note that due to the positive velocity of the occurring waves the boundary conditions are well–defined. In particular and in contrast to [8, 14] no additional maximization problem near at the vertex has to be solved. Moreover, due to (3.1) and the assumption on A we conserve the total flux at each vertex v for all times $t > 0$:

$$\sum_{j \in \delta_v^+} (\partial_t q_j(t) + f_j(\rho_j(a_j+, t))) = \sum_{i \in \delta_v^-} f_i(\rho_i(b_i-, t)).$$

Now, the construction of a solution to the network problem (1.1,3.1,3.2) is as before. In particular, the results of [12] extend to problem (1.1, 3.1, 3.2) on the network $(\mathcal{J}, \mathcal{V})$. It is enough to control the number of waves and interactions: Let $\eta = \min_j(b_j - a_j)$ be the minimum length of a supplier. Since all waves move at positive velocity at most equal to 1, two interactions with vertices of the same wave can happen at most every η units of time. If N is the number of suppliers, than there is at most a multiplication by N every η unit of time, thus we control the number of waves and interactions.

Therefore, for given piecewise constant initial data $\rho^\delta_{j,0}$ on a network, a solution (ρ^δ, q^δ) can be defined by the wave–tracking method up to any time T. Next, we extend Lemma 2.2 to the more general situation of a vertex v above.

3.2 Existence, uniqueness and Lipschitz continuous dependence of a weak solution

We can get again BV estimates:

Lemma 3.1 *Assume a single vertex with incoming arcs $\delta^- = \{1, \dots, m\}$ and outgoing arcs $\delta^+ = \{m+1, \dots, m+n\}$. Furthermore, assume constant states $\rho_{j,0}, j \in \delta^- \cup \delta^+$ at the vertex and consider a discontinuity colliding at time t_0. Denote the new solution at the vertex after the collision by $\bar\rho_j$. Assume no more collision of discontinuities until t^*. Then, for all $t_0 < t < t^*$,*

$$\begin{aligned}&\sum_{j\in\delta^-\cup\delta^+} T.V.(f_j(\rho_j(\cdot,t))) + \sum_{j\in\delta^+} |\partial_t q_j(t)| \\ \le &\sum_{j\in\delta^-\cup\delta^+} T.V.(f_j(\rho_j(\cdot,t_0))) + \sum_{j\in\delta^+} |\partial_t q_j(t_0)|.\end{aligned} \tag{3.3}$$

Proof. The proof is very similar to the proof of Lemma 2.2: The colliding discontinuity has to arrive on an arc $i \in \delta^-$ and we assume $i = 1$. The total variation on the incoming arc $i = 1$ therefore decreases by

$$|f_1(\bar\rho_1) - f_1(\rho_{1,0})| = \sum_{j\in\delta^+} |\alpha_{1j} f_1(\bar\rho_1) - \alpha_{1j} f_1(\rho_{1,0})|.$$

Hence, it suffices to prove that for any fixed outgoing arc $j \in \delta^+$ and for all $t > t_0$ the following inequality holds

$$|\alpha_{1j} f_1(\bar\rho_1) - \alpha_{1j} f_1(\rho_{1,0})| + |\partial_t q_j(t_0)| \ge T.V.(f_j(\rho_j(\cdot,t))) + |\partial_t q_j(t)|. \tag{3.4}$$

Fix $j \in \delta^+$. With the other cases being similar we only discuss the (most interesting) case: Assume

$$\sum_{i\in\delta^-} \alpha_{ij} f_i(\rho_{i,0}) > f_j(\rho_{j,0})$$

and

$$\alpha_{1j} f_1(\bar\rho_1) + \sum_{i,i\neq 1} f_i(\rho_{i,0}) < \mu_j.$$

Then, the queue q_j is decreasing after the collision at time t_0 and we denote again by $\bar{t}$ the time when $q_j(\bar{t})0$. Then for $t < \min\{\bar{t}, t^*\}$ we obtain $T.V.(f_j(\bar{\rho}_j(\cdot, t)) = 0$ and $|\alpha_{1j} f_1(\bar{\rho}_1) - \alpha_{1j} f_1(\rho_{1,0})| + |\partial_t q_j(t_0)| \geq \mu_j - \sum_{i,i\neq 1} \alpha_{ij} f_i(\rho_{i,0}) - \alpha_{1j} f_1(\bar{\rho}_1) = |\partial_t q_j(t)|$. If $\bar{t} < t^*$, then a new discontinuity is generated since the queue q_j empties. By (3.2) we have

$$f_j(\bar{\rho}_j(a_j+, t)) = \sum_{i,i\neq 1} \alpha_{ij} f_j(\rho_{j,0}) + \alpha_{1j} f_1(\bar{\rho}_1)$$

and therefore

$$|\partial_t q(\bar{t})| T.V.(f_j(\rho_j(\cdot, t)))$$

for $t > \bar{t}$. Hence, (3.4) holds for all $t > t_0$. This finishes the proof. □

Therefore, we again obtain the estimate (2.10), where the sum now should run over all arcs and nodes of the network. Moreover, the estimates on $\partial_t q_j$ work also in the same way.

The same arguments as above give existence and uniqueness of a weak solution as well as the Lipschitz continuous dependence on the data in the general case for BV initial data. Finally, Theorem 2.8 holds for a general network.

4 Summary

We have proven existence, uniqueness and Lipschitz continuous dependence of a weak solution to a network model for supply chains. The model consists of a scalar hyperbolic equation governing the dynamics of a supplier and a ordinary differential equation for describing the behavior of the queues. The proof of existence relies on the Front-Tracking approximations and estimates on the total variation.

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