A FLUID DYNAMIC MODEL FOR TELECOMMUNICATION NETWORKS WITH SOURCES AND DESTINATIONS

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Abstract. This paper proposes a macroscopic fluid dynamic model dealing with the flows of information on a telecommunication network with sources and destinations. The model consists of a conservation law for the packets density and a semilinear equation for traffic distributions functions, i.e. functions describing packets paths.

We describe methods to solve Riemann Problems at junctions assigning different traffic distributions functions and two "routing algorithms". Moreover we prove existence of solutions to Cauchy problems for small perturbations of network equilibria.

Key words. data flows on telecommunication networks, sources and destinations, conservation laws, fluid dynamic models

AMS subject classifications. 35L65, 35L67, 90B20

1. Introduction. This paper is concerned with the description and analysis of a macroscopic fluid dynamic model dealing with flows of information on a telecommunication network with sources and destinations. The latter are, respectively, areas from which packets start their travels on the network and areas where they end.

There are various approaches to telecommunication and data networks (see for example [1]), [3], [14], [19], [20]. A first model for telecommunication networks, similar to that introduced recently for car traffic, has been proposed in [9] where two algorithms for dynamics at nodes were considered and existence of solution to Cauchy Problems was proved. The idea is to follow the approach used in [11] for road networks (see also [6], [8], [10], [13], [15], [16], [17]), introducing sources and destinations in the telecommunication model described in [9] and thus taking care of the paths of the packets inside the network.

A telecommunication network consists in a finite collection of transmission lines, modelled by closed intervals of $\mathbb{R}$ connected together by nodes (routers, hubs, switches, etc.). We assume that each node receives and sends information encoded in packets, which can be seen as particles travelling on the network. Taking the Internet network as model, we assume that:

1) Each packet travels on the network with a fixed speed and with assigned final destination;
2) Nodes receive, process and then forward packets. Packets may be lost with a probability increasing with the number of packets to be processed. Each lost packet is sent again.

Since each lost packet is sent again until it reaches next node, looking at macroscopic level, it is assumed that the number of packets is conserved. This leads to a conservation law for the packets density ρ on each line:

$$\rho_t + f(\rho)_x = 0. \tag{1.1}$$

The flux $f(\rho)$ is given by $v \cdot \rho$ where v is the average speed of packets among nodes, derived considering the amount of packets that may be lost.

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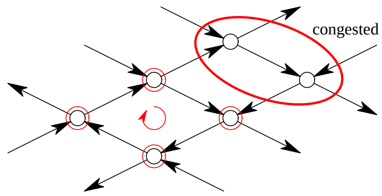

FIG. 1.1. *A possible cycling effect of (RA2).*

Recently, a conservation law model was obtained in [2] for supply chains, which have a dynamics somehow related to our case.

On each transmission line we also consider a vector π describing the traffic types, i.e. the percentages of packets going from a fixed source to a fixed destination. Assuming that packets velocity is independent from the source and the destination, the evolution of π follows a semilinear equation

$$\pi_t + v(\rho)\pi_x = 0, \tag{1.2}$$

hence inside transmission lines the evolution of π is influenced by the average speed of packets.

The aim is then to consider networks in which many lines intersect. Riemann problems at junctions were solved in [9] proposing two different routing algorithms:

(RA1) Packets from incoming lines are sent to outgoing ones according to their final destination (without taking into account possible high loads of outgoing lines);

(RA2) Packets are sent to outgoing lines in order to maximize the flux through the node.

The main differences of the two algorithms are the following. The first one simply sends each packet to the outgoing line which is naturally chosen according to the final destination of the packet itself. The algorithm is blind to possible overloads of some outgoing lines and, by some abuse of notation, is similar to the behavior of a "switch". The second algorithm, on the contrary, send packets to outgoing lines taking into account the loads, and thus possibly redirecting packets. Again by some abuse of notation, this is similar to a "router" behavior.

One of the drawback of the second algorithm is that it does not take into account the global path of packets, therefore leading to possible cycling. For example consider a telecommunication network in which some nodes are congested: if we use (RA2) alone, the packets are not routed towards the congested nodes, and so they can enter in loops (see Figure 1.1). These cyclings are avoided if we consider that the packets originated from a source and with an assigned destination have precise paths inside the network. Such paths are determined by the behaviour at junctions via the coefficients π.

In this paper different distribution traffic functions describing different routing strategies have been considered:

- at a junction the traffic started at source s and with d as final destination, coming from the transmission line i, is routed on an assigned line j;
- at a junction the traffic started at source s and with d as final destination, coming from the transmission line i, is routed on every outgoing lines or on some of them.

The first distribution traffic function has been already analyzed in [11] for road networks using algorithm (RA1), thus we focus on the second one. In particular, we define two ways according to which the traffic at a junction is splitted towards the outgoing lines.
Let us now comment further the differences with the results of [11]. In such paper, only the routing algorithm (RA1) was considered, together with the first choice of distribution traffic functions (which can be seen as a particular case of the second choice.) Since the algorithm (RA1) produces discontinuities in the map from traffic types to fluxes (and densities), a new Riemann solver was introduced, which considers the maximization of a quadratic cost. The latter produces as a drawback more difficulties in analysis and numerics. Finally, the present paper presents a more general approach and, using (RA2), the possibility of solving dynamics at nodes using linear functionals.

Starting from the distribution traffic function, and using the vector π, we assign the traffic distribution matrix, which describes the percentage of packets from an incoming line that are addressed to an outgoing one. Then, we propose methods to solve Riemann Problems considering the routing algorithms (RA1) and (RA2). The key point to construct a solution on the whole network, using a way-front tracking method, is to derive some BV estimates on the piecewise constant approximate solutions, in order to pass to the limit. In the case in which the traffic at junctions is distributed on outgoing lines according to some probabilistic coefficients, estimates on packets density function and on traffic-type functions are derived for the algorithm (RA2) in order to prove existence of solutions to Cauchy problems. More precisely, we prove existence of solutions, locally in time, for perturbations of equilibria.

The paper is organized as follows. Section 2 gives general definition of network. Then, in Section 3, we discuss possible choices of the traffic distribution functions, and how to compute the traffic distribution matrix from the latter functions and the traffic-type function. We describe two routing algorithms in Section 4, giving explicit unique solutions to Riemann problems. Finally, Section 5 provides the needed estimates for constructing solutions to Cauchy problems.

2. Basic definitions. We consider a telecommunication network that is a finite collection of transmission lines connected together by nodes, some of which are sources and destinations. Formally we introduce the following definition:

Definition 2.1. *A telecommunication network is given by a 7-tuple* $(N, \mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R})$ *where*

Cardinality *N is the cardinality of the network, i.e. the number of lines in the network;*

Lines *$\mathcal{I}$ is the collection of lines, modelled by intervals $I_i = [a_i, b_i] \subseteq \mathbb{R}, i = 1, ..., N$;*

Fluxes *$\mathcal{F}$ is the collection of flux functions $f_i : [0, \rho_i^{\max}] \mapsto \mathbb{R}$, $i = 1, ..., N$;*

Nodes *$\mathcal{J}$ is a collection of subsets of $\{\pm 1, ..., \pm N\}$ representing nodes. If $j \in J \in \mathcal{J}$, then the transmission line $I_{|j|}$ is crossing at J as incoming line (i.e. at point b_i) if $j > 0$ and as outgoing line (i.e. at point a_i) if $j < 0$. For each junction $J \in \mathcal{J}$, we indicate by $\mathrm{Inc}(J)$ the set of incoming lines, that are I_i's such that $i \in J$, while by $\mathrm{Out}(J)$ the set of outgoing lines, that are I_i's such that $-i \in J$. We assume that each line is incoming for (at most) one node and outgoing for (at most) one node;*

Sources *$\mathcal{S}$ is the subset of $\{1, ..., N\}$ representing lines starting from traffic sources. Thus, $j \in \mathcal{S}$ if and only if j is not outgoing for any node. We assume that $\mathcal{S} \neq \emptyset$;*

Destinations $\mathcal{D}$ *is the subset of* $\{1, ..., N\}$ *representing lines leading to traffic destinations, Thus,* $j \in \mathcal{D}$ *if and only if* j *is not incoming for any node. We assume that* $\mathcal{D} \neq \emptyset$*;*

Traffic distribution functions $\mathcal{R}$ *is a finite collection of functions* $r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \to \mathrm{Out}(J)$*. For every* J*,* $r_J(i, s, d)$ *indicates the outgoing direction of traffic that started at source* s*, has* d *as final destination and reached* J *from the incoming road* i*. (We will consider also the case of* r_J *multivalued.)*

One usually assumed that the network is connected. However, this is not strictly necessary to develop our theory.

2.1. Dynamics on lines. Following [9], we recall the model used to define the dynamics of packet densities along lines. We make the following hypothesis:

(H1) Lines are composed of consecutive processors N_k, which receive and send packets. The number of packets at N_k is indicated by $R_k \in [0, R_{max}]$;

(H2) There are two time-scales: Δt_0, which represents the physical travel time of a single packet from node to node (assumed to be independent of the node for simplicity); T representing the processing time, during which each processor tries to operate the transmission of a given packet;

(H3) Each processor N_k tries to send all packets R_k at the same time. Packets are lost according to a loss probability function $p : [0, R_{max}] \to [0, 1]$, computed at R_{k+1}, and lost packets are sent again for a time slot of length T.

The aim is to determine the fluxes on the network. Since the packet transmission velocity on the line is assumed constant, it is possible to compute an average velocity function and thus an average flux function.

Let us focus on two consecutive nodes N_k and N_{k+1}, assume a static situation, i.e. R_k and R_{k+1} are constant, and call δ the distance between the nodes. During a processing time slot of length T the following happens. All packets R_k are sent a first time: $(1 - p(R_{k+1}))\, R_k$ are sent successfully and $p(R_{k+1})\, R_k$ are lost. At the second attempt, of the lost packets $p(R_{k+1})\, R_k$, $(1-p(R_{k+1})\, p(R_{k+1})\, R_k$ are sent successfully and $p^2(R_{k+1})\, R_k$ are lost and so on.
Let us indicate by Δt_{av} the average transmission time of packets, by $\bar{v} = \frac{\delta}{\Delta t_0}$ the packet velocity without losses and $v = \frac{\delta}{\Delta t_{av}}$ the average packets velocity. Then, we can compute:

$$\Delta t_{av} = \sum_{n=1}^{M} n \Delta t_0 (1 - p(R_{k+1})) p^{n-1}(R_{k+1})$$

where $M = [T/\Delta t_0]$ (here $[\cdot]$ indicates the floor function) represents the number of attempts of sending a packet. We make a further assumption:

(H4) The number of packets not transmitted for a whole processing time slot is negligible.

The hypothesis (H4) corresponds to assume $\Delta t_0 << T$ or, equivalently, $M \sim +\infty$. Making the identification, $M = +\infty$, we get:

$$\Delta t_{av} = \sum_{n=1}^{+\infty} n \Delta t_0 (1 - p(R_{k+1})) p^{n-1}(R_{k+1}) = \frac{\Delta t_0}{1 - p(R_{k+1})},$$

and

$$v = \frac{\delta}{\Delta t_{av}} = \frac{\delta}{\Delta t_0}(1 - p(R_{k+1})) = \bar{v}(1 - p(R_{k+1})). \tag{2.1}$$

Let us call now ρ the averaged density and ρ_{max} its maximum. We can interpret the function p as a function of ρ and, using (2.1), determine the corresponding flux function, given by the averaged density times the average velocity. It is reasonable to assume that the probability loss function is null for some interval, which is a right neighborhood of zero. This means that at low densities no packet is lost. Then p should be increasing, reaching the value 1 at the maximal density, the situation of complete stuck. A possible choice of the probability loss function is the following:

$$p(\rho) = \begin{cases} 0, & 0 \le \rho \le \sigma, \\ \frac{\rho_{max}(\rho-\sigma)}{\rho(\rho_{max}-\sigma)}, & \sigma \le \rho \le \rho_{\max}, \end{cases}$$

then, it follows that

$$f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}\sigma(\rho_{max}-\rho)}{\rho_{max}-\sigma}, & \sigma \le \rho \le \rho_{\max}. \end{cases} \tag{2.2}$$

Setting, for simplicity $\rho_{max} = 1$ and $\sigma = \frac{1}{2}$, we get the simple "tent" function of Figure 2.1. To simplify the treatment of the corresponding conservation laws, we will

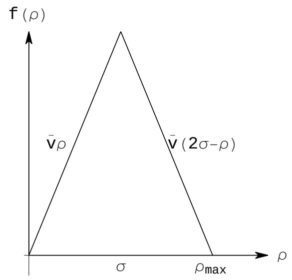

FIG. 2.1. *Example of flux function.*

assume the following:

(F) Setting $\rho_{max} = 1$, on each line the flux $f_i : [0,1] \to R$ is concave, $f(0) = f(1) = 0$ and there exists a unique maximum point $\sigma \in]0,1[$.

Notice that the flux of Figure 2.1 or, more generally, the flux given in (2.2) satisfies the assumption (F).

2.2. Dynamics on the network. On each transmission line I_i we consider the evolution equation

$$\partial_t \rho_i + \partial_x f_i(\rho_i) = 0, \tag{2.3}$$

where we use the assumption (F). Therefore, the network load evolution is described by a finite set of functions $\rho_i : [0,+\infty[\times I_i \mapsto [0, \rho_i^{\max}]$.

On each transmission line I_i we want ρ_i to be a weak entropic solution of (2.3), that is for every function $\varphi : [0,+\infty[\times I_i \to \mathbb{R}$ smooth, positive with compact support on $]0,+\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left(\rho_i \frac{\partial \varphi}{\partial t} + f_i(\rho_i) \frac{\partial \varphi}{\partial x} \right) dx dt = 0, \tag{2.4}$$

and for every $k \in \mathbb{R}$ and every $\tilde{\varphi} : [0, +\infty[\times I_i \rightarrow \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left(|\rho_i - k| \frac{\partial \tilde{\varphi}}{\partial t} + sgn(\rho_i - k) \left(f_i(\rho_i) - f_i(k) \right) \frac{\partial \tilde{\varphi}}{\partial x} \right) dxdt \geq 0. \tag{2.5}$$

For each $i \in \mathcal{S}$ (resp. $i \in \mathcal{D}$) we need an inflow function (resp. outflow), thus consider measurable functions $\psi_i : [0, +\infty[\rightarrow [0, \rho_i^{\max}]$. Then the corresponding functions ρ_i must verify the boundary condition $\rho_i(t, a_i) = \psi_i(t)$ (resp. $\rho_i(t, b_i) = \psi_i(t)$) in the sense of [4].

Moreover, inside each line I_i we define a traffic-type function π_i, which measures the portion of the whole density coming from each source and travelling towards each destination:

Definition 2.2. *A traffic-type function on a line I_i is a function*

$$\pi_i : [0, \infty[\times [a_i, b_i] \times \mathcal{S} \times \mathcal{D} \mapsto [0, 1]$$

such that, for every $t \in [0, \infty[$ and $x \in [a_i, b_i]$

$$\sum_{s \in \mathcal{S}, d \in \mathcal{D}} \pi_i(t, x, s, d) = 1.$$

In other words, $\pi_i(t, x, s, d)$ specifies the fraction of the density $\rho_i(t, x)$ that started from source s and is moving towards the final destination d.

We assumed, on the discrete model, that a FIFO policy is used at nodes. Then it is natural that the averaged velocity, obtained in the limit procedure, is independent from the original sources of packets and their final destinations. In other words, we make the following hypothesis:

(H5) On each line I_i, the average velocity of packets depends only on the value of the density ρ_i and not on the values of the traffic-type function π_i.

As a consequence of hypothesis (H5), we have the following. If $x(t)$ denotes a trajectory of a packet inside the line I_i, then we get

$$\pi_i(t, x(t), s, d) = const. \tag{2.6}$$

In fact, consider the packets that at time t are in position $x(t)$. All such packets have the same velocity by (H5), thus their trajectories coincide, independently of their sources and destinations. In other words at a time $t' > t$ all packets will be in position $x(t')$. Then the fractions of the density, expressed by π, are the same at $(t, x(t))$ and at $(t', x(t'))$.

Taking the total differential with respect to the time of (2.6), we deduce the semilinear equation

$$\partial_t \pi_i(t, x, s, d) + \partial_x \pi_i(t, x, s, d) \cdot v_i(\rho_i(t, x)) = 0. \tag{2.7}$$

This equation is coupled with equation (2.3) on each line I_i. More precisely, equation (2.7) depends on the solution of (2.3), while in turn at junctions the values of π_i will determine the traffic distribution on outgoing lines as explained below.

For simplicity and without loss of generality, we assume from now on that the fluxes f_i are all the same and we indicate them with f. Thus, the model for a single transmission line, consists in the system of equations:

$$\begin{cases} \rho_t + f(\rho)_x = 0, \\ \pi_t + \pi_x \cdot v(\rho) = 0. \end{cases}$$

To treat the evolution at junction, let us introduce some notations. Fix a junction J with n incoming transmission lines, say $I_1, ..., I_n$, and m outgoing transmission lines, say $I_{n+1}, ..., I_{n+m}$. A weak solution at J is a collection of functions $\rho_l : [0, +\infty[\times I_l \mapsto \mathbb{R}, l = 1, ..., n+m$, such that

$$\sum_{l=1}^{n+m} \left(\int_0^{+\infty} \int_{a_l}^{b_l} \left(\rho_l \frac{\partial \varphi_l}{\partial t} + f(\rho_l) \frac{\partial \varphi_l}{\partial x} \right) dxdt \right) = 0, \tag{2.8}$$

for every $\varphi_l, l = 1, ..., n+m$, smooth having compact support in $]0, +\infty[\times]a_l, b_l]$ for $l = 1, ..., n$ (incoming transmission lines) and in $]0, +\infty[\times [a_l, b_l[$ for $l = n+1, ..., n+m$ (outgoing transmission lines), that are also smooth across the junction, i.e.

$$\varphi_i(\cdot, b_i) = \varphi_j(\cdot, a_j), \quad \frac{\partial \varphi_i}{\partial x}(\cdot, b_i) = \frac{\partial \varphi_j}{\partial x}(\cdot, a_j), \quad i = 1, ..., n, j = n+1, ..., n+m.$$

Remark 2.3. *Let $\rho = (\rho_1, ..., \rho_{n+m})$ be a weak solution at the junction J such that each $x \to \rho_i(t, x)$ has bounded variation. We can deduce that ρ satisfies the Rankine-Hugoniot condition at J, namely*

$$\sum_{i=1}^{n} f(\rho_i(t, b_i-)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, a_j+)),$$

for almost every $t > 0$.

For a scalar conservation law a Riemann problem is a Cauchy problem for an initial data of Heavyside type, that is piecewise constant with only one discontinuity. One looks for centered solutions, i.e. $\rho(t, x) = \phi(\frac{x}{t})$ formed by simple waves, which are the building blocks to construct solutions to the Cauchy problem via wave-front tracking algorithm. These solutions are formed by continuous waves called rarefactions and by travelling discontinuities called shocks. The speed of waves are related to the values of f', see [5], [7], [18].

Analogously, we call Riemann problem for a junction the Cauchy problem corresponding to an initial data $\rho_{1,0}, ..., \rho_{n+m,0} \in [0, 1]$, and $\pi_1^{s,d}, ..., \pi_{n+m}^{s,d} \in [0, 1]$ which are constant on each transmission line.

Definition 2.4. *A Riemann Solver (RS) for the junction J is a map that associates to Riemann data $\rho_0 = (\rho_{1,0}, \ldots, \rho_{n+m,0})$ and $\Pi_0 = (\pi_{1,0}, \ldots, \pi_{n+m,0})$ at J the vectors $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_{n+m})$ and $\hat{\Pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_{n+m})$ so that the solution on an incoming transmission line I_i, $i = 1, \ldots, n$, is given by the wave $(\rho_{i,0}, \hat{\rho}_i)$ and on an outgoing one I_j, $j = n+1, \ldots, n+m$, is given by the waves$(\hat{\rho}_j, \rho_{j,0})$ and $(\hat{\pi}_j, \pi_{j,0})$. We require the following consistency condition:*

(CC) $RS(RS(\rho_0, \Pi_0)) = RS(\rho_0, \Pi_0)$.

We will define a RS at a junction in next sections. Once a Riemann solver is defined and the solution of the Riemann Problem is obtained, we can define admissible solutions at junctions.

Definition 2.5. *Assume a Riemann solver RS is assigned. Let $\rho = (\rho_1, \ldots, \rho_{n+m})$ and $\Pi = (\pi_1, \ldots, \pi_{n+m})$ be such that $\rho_i(t, \cdot)$ and $\pi_i(t, \cdot)$ are of bounded variation for every $t \geq 0$. Then (ρ, Π) is an admissible weak solution of (1.1) related to RS at the junction J if and only if the following properties hold:*

(i) *ρ is a weak solution at junction J;*

(ii) *Π is a weak solution at junction J;*

(iii) for almost every t setting

$$\rho_J(t) = (\rho_1(\cdot, b_1-), \dots, \rho_n(\cdot, b_n-), \rho_{n+1}(\cdot, a_{n+1}+), \dots, \rho_{n+m}(\cdot, a_{n+m}+)),$$
$$\Pi_J(t) = (\pi_1(\cdot, b_1-), \dots, \pi_n(\cdot, b_n-), \pi_{n+1}(\cdot, a_{n+1}+), \dots, \pi_{n+m}(\cdot, a_{n+m}+))$$

we have

$$RS(\rho_J(t), \Pi_J(t)) = (\rho_J(t), \Pi_J(t)).$$

Given an admissible network (see [11]) we have to specify how to define a solution.

DEFINITION 2.6. *Consider an admissible network $(N, \mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R})$. A set of Initial-Boundary Conditions (briefly IBC) is given assigning measurable functions $\bar{\rho}_i : I_i \mapsto [0, \rho_i^{\max}], \bar{\pi}_i : [a_i, b_i] \times \mathcal{S} \times \mathcal{D} \mapsto [0,1], i = 1, ..., N$ and measurable functions $\psi_i : [0, +\infty[\mapsto [0, \rho_i^{\max}], i \in \mathcal{S} \cup \mathcal{D}$ and $\vartheta_{i,j} : [0, +\infty[\mapsto [0,1], i \in \mathcal{S}, j \in \mathcal{D}$ with the property that $\sum_j \vartheta_{i,j}(t) = 1$.*

DEFINITION 2.7. *Consider an admissible network $(N, \mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R})$ and a set of IBC. A set of functions $\rho = (\rho_1, ..., \rho_N)$ with $\rho_i : [0, +\infty[\times I_i \mapsto [0, \rho_i^{\max}]$ continuous as functions from $[0, +\infty[$ into L^1, and $\Pi = (\pi_1, ..., \pi_N)$ with $\pi_i : [0, +\infty[\times I_i \times \mathcal{S} \times \mathcal{D} \mapsto [0,1]$, continuous as functions from $[0, +\infty[$ into L^1 for every $s \in \mathcal{S}, d \in \mathcal{D}$, is an admissible solution if the following holds. Each ρ_i is a weak entropic solution to (2.3) on $I_i, \rho_i(0,x) = \bar{\rho}_i(x)$ for almost every $x \in [a_i, b_i], \rho_i(t, a_i) = \psi_i(t)$ if $i \in S$ and $\rho_i(t, b_i) = \psi_i(t)$ if $i \in \mathcal{D}$ in the sense of [4]. Each π_i is a weak solution to the corresponding equation (2.7), $\pi_i(0,x) = \bar{\pi}_i(x)$ for almost every $x \in [a_i, b_i]$ and for every $i \in \mathcal{S}, j \in \mathcal{D}$ $\pi_i^{i,j}(t, a_i) = \vartheta_{i,j}$ in the sense of [4]. Finally at each junction (ρ, Π) is a weak solution and is an admissible weak solution in case of bounded variation.*

3. Traffic distribution at junctions. Consider a junction J in which there are n transmission lines with incoming traffic and m transmission lines with outgoing traffic.

We denote with $\rho_i(t,x), i = 1, ..., n$ and $\rho_j(t,x), j = n+1, ..., n+m$ the traffic densities, respectively, on the incoming transmission lines and on the outgoing ones and by $(\rho_{1,0}, .., \rho_{n+m,0})$ the initial datum.

Define $\gamma_i^{\max}$ and $\gamma_j^{\max}$ as follows:

$$\gamma_i^{\max} = \begin{cases} f(\rho_{i,0}), & \text{if } \rho_{i,0} \in [0, \sigma], \\ f(\sigma), & \text{if } \rho_{i,0} \in]\sigma, 1], \end{cases} \quad i = 1, ..., n, \tag{3.1}$$

and

$$\gamma_j^{\max} = \begin{cases} f(\sigma), & \text{if } \rho_{j,0} \in [0, \sigma], \\ f(\rho_{j,0}), & \text{if } \rho_{j,0} \in]\sigma, 1], \end{cases} \quad j = n+1, ..., n+m. \tag{3.2}$$

The quantities $\gamma_i^{\max}$ and $\gamma_j^{\max}$ represent the maximum flux that can be obtained by a single wave solution on each transmission line. Finally denote with

$$\Omega_i = [0, \gamma_i^{\max}], i = 1, ..., n,$$
$$\Omega_j = [0, \gamma_j^{\max}], j = n+1, ..., n+m,$$

and with $\hat{\gamma}_{inc} = (f(\hat{\rho}_i), ..., f(\hat{\rho}_n)), \hat{\gamma}_{out} = (f(\hat{\rho}_{n+1}), ..., f(\hat{\rho}_{n+m}))$ where $\hat{\rho} = (\hat{\rho}_1, ..., \hat{\rho}_{n+m})$ is the solution of the Riemann Problem at the junction.

Now, we discuss some possible choices for the traffic distribution function:

1) $r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \to \mathrm{Out}(J)$;

2) $r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \hookrightarrow \mathrm{Out}(J)$, i.e. r_J is a multifunction.

If r_J is of type 1), then each packet has a deterministic route, it means that, at the junction J, the traffic that started at source s and has d as final destination, coming from the transmission line i, is routed on an assigned line j $(r_J(i,s,d) = j)$.

Instead if r_J is of type 2), at the junction J, the traffic with source s and destination d coming from a line i is routed on every line $I_j \in \mathrm{Out}(J)$ or on some lines $I_j \in \mathrm{Out}(J)$. We can define $r_J(i,s,d)$ in two different ways:

2a) $r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \hookrightarrow \mathrm{Out}(J)$,
$r_j(i,s,d) \subseteq \mathrm{Out}(J)$;

2b) $r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \to [0,1]^{\mathrm{Out}(J)}$,
$r_J(i,s,d) = (\alpha_J^{i,s,d,n+1}, ..., \alpha_J^{i,s,d,n+m})$

with $0 \leq \alpha_J^{i,s,d,j} \leq 1, j \in \{n+1,...,n+m\}, \sum_{j=n+1}^{n+m} \alpha_J^{i,s,d,j} = 1.$

In case 2a) we have to specify in which way the traffic at junction J is splitted towards the outgoing lines.

The definition 2b) means that, at the junction J, the traffic with source s and destination d coming from line I_i is routed on the outgoing line $j, j = n+1,...,n+m$ with probability $\alpha_J^{i,s,d,j}$.

Let us analyze how the distribution matrix A is constructed using π and r_J.

Definition 3.1. *A distribution matrix is a matrix*

$$A \doteq \{\alpha_{j,i}\}_{j=n+1,...,n+m, i=1,...,n} \in \mathbb{R}^{m \times n}$$

such that

$$0 < \alpha_{j,i} < 1, \qquad \sum_{j=n+1}^{n+m} \alpha_{j,i} = 1,$$

for each $i = 1,...,n$ and $j = n+1,...,n+m$, where $\alpha_{j,i}$ is the percentage of packets arriving from the i-th incoming transmission line that take the j-th outgoing transmission line.

In case 1) we can define the matrix A in the following way. Fix a time t and assume that for all $i \in \mathrm{Inc}(J), s \in \mathcal{S}$ and $d \in \mathcal{D}, \pi_i(t,\cdot,s,d)$ admits a limit at the junction J, i.e left limit at b_i. For $i \in \{1,...,n\}, j \in \{n+1,...,n+m\}$, we set

$$\alpha_{j,i} = \sum_{\substack{s \in \mathcal{S}, d \in \mathcal{D}, \\ r_J(i,s,d) = j}} \pi_i(t, b_i-, s, d).$$

The fluxes $f_i(\rho_i)$ to be consistent with the traffic-type functions must satisfy the following relation:

$$f_j(\rho_j(\cdot, a_j+)) = \sum_{i=1}^{n} \alpha_{j,i} f_i(\rho_i(\cdot, b_i-)),$$

for each $j = n+1,...,n+m$.

Let us analyze how to define the matrix A in the case 2a). We may assign $\varphi(i,s,d) \in r_J(i,s,d)$ and set

$$\alpha_{j,i} = \sum_{\substack{s \in \mathcal{S}, d \in \mathcal{D}, \\ i : \varphi(i,s,d) = j}} \pi_i(t, b_i-, s, d),$$

$$\alpha_{j,i} = 0, \text{ if } j \notin r_J(i,s,d).$$

Example 3.2. *Fix a junction J with two incoming lines $\{1,2\}$ and two outgoing lines $\{3,4\}$ and suppose that $r_J(1,s,d)=\{3,4\}$ and $r_J(2,s,d)=\{3\}$. Since $\alpha_{4,2}=0$, we have $\alpha_{3,2}=1$. The coefficients $\alpha_{3,1}$ and $\alpha_{4,1}$ can assume the following values:*

$$\begin{cases} \alpha_{3,1}=0, \\ \alpha_{4,1}=1, \end{cases} or \begin{cases} \alpha_{3,1}=1, \\ \alpha_{4,1}=0. \end{cases}$$

We get a finite number of possible distribution matrices A :

$$A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Remark 3.3. *This model proposes an exclusive strategy, in fact all packets flow at the junction is routed from line 1 to line 3 or to line 4.*

However, it is more natural to assign a flexible strategy defining a set of admissible matrices A in the following way

$$\mathcal{A}=\left\{\begin{array}{c} A:\exists\alpha_J^{i,s,d,j}\in[0,1], \sum_{j=n+1}^{n+m}\alpha_J^{i,s,d,j}=1, \alpha_J^{i,s,d,j}=0, \text{ if } j\notin r_J(i,s,d): \\ \alpha_{j,i}=\sum_{\substack{s\in\mathcal{S},d\in\mathcal{D}, \\ j\in r_J(i,s,d)}}\pi_i(t,b_i-,s,d)\alpha_J^{i,s,d,j} \end{array}\right\}$$

Finally, we treat now the case 2b).

In this case the matrix A is unique and is defined by

$$\alpha_{j,i}=\sum_{s\in\mathcal{S},d\in\mathcal{D}}\pi_i(t,b_i-,s,d)\alpha_J^{i,s,d,j}. \tag{3.3}$$

4. Riemann solvers at junctions. In this section we define solutions to Riemann problems at junctions, since this is the basic ingredient to construct solution to Cauchy problems via wave-front tracking algorithm.

We describe two different Riemann solvers at a junction that represent two different routing algorithms:

(RA1) We assume that
- (A) the traffic from incoming transmission lines is distributed on outgoing transmission lines according to fixed coefficients;
- (B) respecting (A) the router chooses to send packets in order to maximize fluxes (i.e., the number of packets which are processed).

(RA2) We assume that the number of packets through the junction is maximized both over incoming and outgoing lines.

Remark 4.1. *In what follows we analyze the case in which the traffic distribution function is of type 2). The case 1) has been considered in [11] using the following rule:*

(RGP) *We assume that*

(A) *the traffic from incoming transmission lines is distributed on outgoing transmission lines according to fixed coefficients;*

(B) *respecting (A) the router chooses to send packets in order to maximize*

$$c_2\sum_{i=1}^{n}f_i(\rho_i(\cdot,b_i-))-c_1[dist((f_1(\rho_1(\cdot,b_1-)),...,f_n(\rho_n(\cdot,b_n-))),r)]^2$$

subject to

$$f_j(\rho_j(\cdot, a_j+)) = \sum_{i=1}^{n} \alpha_{j,i} f_i(\rho_i(\cdot, b_i-)), \quad \textit{for each } j = n+1, ..., n+m,$$

where c_1 *and* c_2 *are strictly positive constants, and* $dist(\cdot, r)$ *denotes the Euclidean distance in* $\mathbb{R}^n$ *from the line* r*, which is given by*

$$\begin{cases} \gamma_2 = p_1 \gamma_1, \\ \vdots \\ \gamma_n = p_{n-1}\gamma_{n-1}. \end{cases}$$

and $(p_1, ..., p_{n-1})$ *determine a "level of priority" at the junctions of incoming lines. This maximization procedure takes into account priorities over incoming roads and ensures continuity of solutions respect to the coefficients* π.

4.1. Algorithm (RA1). We have to distinguish case 2a) and 2b).

In case 2a) in order to solve the Riemann Problem at the junction we have to prove that the admissible region is convex. First we prove the following lemma

LEMMA 4.2. *The set* $\mathcal{A}$ *is convex.*

Proof. Let us consider a convex combination $\lambda A_1 + (1-\lambda) A_2$ with $\lambda \in [0,1]$, $A_1, A_2 \in \mathcal{A}$. We have

$$(\lambda A_1 + (1-\lambda)\mathcal{A}_2)_{i,j} = \lambda \sum_{\substack{s\in\mathcal{S}, d\in\mathcal{D}, \\ j\in r_J(i,s,d)}} \pi_i \alpha_{J,1}^{i,s,d,j} + (1-\lambda) \sum_{\substack{s\in\mathcal{S}, d\in\mathcal{D}, \\ j\in r_J(i,s,d)}} \pi_i \alpha_{J,2}^{i,s,d,j} =$$

$$\sum_{\substack{s\in\mathcal{S}, d\in\mathcal{D}, \\ j\in r_J(i,s,d)}} \pi_i (\lambda \alpha_{J,1}^{i,s,d,j} + (1-\lambda)\alpha_{J,2}^{i,s,d,j}) = \sum_{\substack{s\in\mathcal{S}, d\in\mathcal{D}, \\ j\in r_J(i,s,d)}} \pi_i \hat{\alpha}_J^{i,s,d,j},$$

with $\hat{\alpha}_J^{i,s,d,j} \in [0,1]$. Moreover

$$\sum_{j=n+1}^{n+m} \hat{\alpha}_J^{i,s,d,j} = \sum_{j=n+1}^{n+m} (\lambda \alpha_{J,1}^{i,s,d,j} + (1-\lambda)\alpha_{J,2}^{i,s,d,j}) = \lambda \sum_{j=n+1}^{n+m} \alpha_{J,1}^{i,s,d,j} + (1-\lambda) \sum_{j=n+1}^{n+m} \alpha_{J,2}^{i,s,d,j} = 1,$$

then $\lambda A_1 + (1-\lambda)A_2 \in \mathcal{A}$. □

Now recall that the admissible region is given by:

$$\Omega_{adm} = \{\hat{\gamma} : \hat{\gamma} \in \Omega_1 \times ... \times \Omega_n, \exists A \in \mathcal{A} \; t.c. A\hat{\gamma} \in \Omega_{n+1} \times ... \times \Omega_{n+m}\}.$$

We can prove that this region is convex at least for the case of junctions with two incoming and two outgoing lines, more precisely we have:

LEMMA 4.3. *Fix a junction* J *with* $n = 2$ *incoming lines and* $m = 2$ *outgoing ones and assume that there is a unique source and a unique destination. Then the set* Ω_{adm} *is convex.*

Proof. We have to consider the following cases:

i) $r_J(1,s,d) = 3$ and $r_J(2,s,d) = 3$;
ii) $r_J(1,s,d) = 3$ and $r_J(2,s,d) = 4$;
iii) $r_J(1,s,d) = \{3,4\}$ and $r_J(2,s,d) = 4$;
iv) $r_J(1,s,d) = \{3,4\}$ and $r_J(2,s,d) = \{3,4\}$.

All other cases can be obtained by relabelling lines. Cases i) and ii) are immediate, since $\hat{\gamma} \in \Omega_1 \times \Omega_2$ satisfies $\hat{\gamma} \in \Omega_{adm}$ if and only if $\hat{\gamma}_1 + \hat{\gamma}_2 \leq \gamma_3^{max}$ (case i)) or $\hat{\gamma}_1 \leq \gamma_3^{max}$ and $\hat{\gamma}_2 \leq \gamma_4^{max}$ (case ii).)
Consider now case iii). Then $A\hat{\gamma}$, $A \in \mathcal{A}$, is the segment joining the point $(\hat{\gamma}_1, \hat{\gamma}_2)$ to the point $(0, \hat{\gamma}_1 + \hat{\gamma}_2)$. Thus $\hat{\gamma} \in \Omega_1 \times \Omega_2$ satisfies $\hat{\gamma} \in \Omega_{adm}$ if and only if $\hat{\gamma}_1 + \hat{\gamma}_2 \leq \gamma_3^{max} + \gamma_4^{max}$ and $\hat{\gamma}_2 \leq \gamma_3^{max}$.
Finally, assume case iv) holds true. Then $A\hat{\gamma}$, $A \in \mathcal{A}$, is the segment joining the point $(\hat{\gamma}_1 + \hat{\gamma}_2, 0)$ to the point $(0, \hat{\gamma}_1 + \hat{\gamma}_2)$. Thus $\hat{\gamma} \in \Omega_1 \times \Omega_2$ satisfies $\hat{\gamma} \in \Omega_{adm}$ if and only if $\hat{\gamma}_1 + \hat{\gamma}_2 \leq \gamma_3^{max} + \gamma_4^{max}$. □

If the region Ω_{adm} is convex than rules (A) and (B) amount to the Linear Programming problem:

$$\max_{\hat{\gamma} \in \Omega_{adm}} (\hat{\gamma}_1 + \hat{\gamma}_2).$$

This problem has clearly a solution, which may not be unique.

Let us consider the case 2b). We need some more notations.

DEFINITION 4.4. *Let $\tau : [0,1] \ \rightarrow [0,1]$ be the map such that:*

1. *$f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0,1]$;*
2. *$\tau(\rho) \neq \rho$ for every $\rho \in [0,1] \backslash \{\sigma\}$.*

Clearly, τ is well defined and satisfies

$$\begin{aligned} 0 \leq \rho \leq \sigma \Leftrightarrow \sigma \leq \tau(\rho) \leq 1, \\ \sigma \leq \rho \leq 1 \Leftrightarrow 0 \leq \tau(\rho) \leq \sigma. \end{aligned}$$

To state the main result of this section we need some assumption on the matrix A (satisfied under generic conditions for $m = n$). Let $\{e_1, ..., e_n\}$ be the canonical basis of $\mathbb{R}^n$ and for every subset $V \subset \mathbb{R}^n$ indicate by $V^{\perp}$ its orthogonal. Define for every $i = 1, ..., n$, $H_i = \{e_i\}^{\perp}$, i.e. the coordinate hyperplane orthogonal to e_i and for every $j = n+1, ..., n+m$ let $\alpha_j = \{\alpha_{j1}, ..., \alpha_{jn}\} \in \mathbb{R}^n$ and define $H_j = \{\alpha_j\}^{\perp}$. Let $\mathcal{K}$ be the set of indices $k = (k_1, ..., k_l), 1 \leq l \leq n-1$, such that $0 \leq k_1 < k_2 < ... < k_l \leq n+m$ and for every $k \in \mathcal{K}$ set $H_k = \bigcap_{h=1}^{l} H_h$. Letting $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^n$, we assume

(C) for every $k \in \mathcal{K}$, $\mathbf{1} \notin H_k^{\perp}$.

In case 2b) the following result holds

THEOREM 4.5. *(Theorem 3.1 in [6] and 3.2 in [11]) Let $(N, \mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R})$ be an admissible network and J a junction with n incoming lines and m outgoing ones. Assume that the flux $f : [0,1] \ \rightarrow \mathbb{R}$ satisfies (F) and the matrix A satisfies condition (C). For every $\rho_{1,0}, ..., \rho_{n+m,0} \in [0,1]$, and for every $\pi_1^{s,d}, ...\pi_{n+m}^{s,d} \in [0,1]$, there exists densities $\hat{\rho}_1, ..., \hat{\rho}_{n+m}$ and a unique admissible centered weak solution, $\rho = (\rho_1, ..., \rho_{n+m})$ at the junction J such that*

$$\begin{aligned} \rho_1(0, \cdot) \equiv \rho_{1,0}, ..., \rho_{n+m}(0, \cdot) \equiv \rho_{n+m,0}, \\ \pi^1(0, \cdot s, d) = \pi_1^{s,d}, ..., \pi^{n+m}(0, \cdot, s, d) = \pi_{n+m}^{s,d}, (s \in \mathcal{S}, d \in \mathcal{D}). \end{aligned}$$

We have

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1] , & \text{if } 0 \leq \rho_{i,0} \leq \sigma, \\ [\sigma, 1] , & \text{if } \sigma \leq \rho_{i,0} \leq 1, \end{cases} \quad i = 1, ..., n, \tag{4.1}$$

$$\hat{\rho}_j \in \begin{cases} [0, \sigma], & \text{if } 0 \leq \rho_{j,0} \leq \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[, & \text{if } \sigma \leq \rho_{j,0} \leq 1, \end{cases} \quad j = n+1, ..., n+m, \tag{4.2}$$

and on each incoming line I_i, $i = 1, ..., n$, the solution consists of the single wave $(\rho_{i,0}, \hat{\rho}_i)$, while on each outgoing line I_j, $j = n+1, ..., n+m$, the solution consists of the single wave $(\hat{\rho}_j, \rho_{j,0})$. Moreover $\hat{\pi}_i(t, \cdot, s, d) = \pi_i^{s,d}$ for every $t \geq 0, i \in \{1, ..., n\}, s \in \mathcal{S}, d \in \mathcal{D}$ and

$$\hat{\pi}_j(t, a_j+, s, d) = \frac{\displaystyle\sum_{i=1}^{n} \alpha_J^{i,s,d,j} \pi_i^{s,d}(t, b_i-, s, d) f(\hat{\rho}_i)}{f(\hat{\rho}_j)}$$

for every $t \geq 0, j \in \{n+1, ..., n+m\}, s \in \mathcal{S}, d \in \mathcal{D}$.

4.2. Algorithm (RA2). To solve Riemann problems according to (RA2) we need some additional parameters called priority and traffic distribution parameters. For simplicity of exposition, consider, first a junction J in which there are two transmission lines with incoming traffic and two transmission lines with outgoing traffic. In this case we have only one priority parameter $q \in]0, 1[$ and one traffic distribution parameter $\alpha \in]0, 1[$. We denote with $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ and $(\pi_{1,0}^{s,d}, \pi_{2,0}^{s,d}, \pi_{3,0}^{s,d}, \pi_{4,0}^{s,d})$ the initial data.

In order to maximize the number of packets through the junction over incoming and outgoing lines we define

$$\Gamma = \min \{\Gamma_{in}^{\max}, \Gamma_{out}^{\max}\},$$

where $\Gamma_{in}^{\max} = \gamma_1^{\max} + \gamma_2^{\max}$ and $\Gamma_{out}^{\max} = \gamma_3^{\max} + \gamma_4^{\max}$. Thus we want to have Γ as flux through the junction.

One easily see that to solve the Riemann problem, it is enough to determine the fluxes $\hat{\gamma}_i = f(\hat{\rho}_i), i = 1, 2$. In fact, to have simple waves with the appropriate velocities, i.e. negative on incoming lines and positive on outgoing ones, we get the constraints (4.1), (4.2). Observe that we compute $\hat{\gamma}_i = f(\hat{\rho}_i), i = 1, 2$ without taking into account the type of traffic distribution function.

We have to distinguish two cases:

I $\Gamma_{in}^{\max} = \Gamma$,

II $\Gamma_{in}^{\max} > \Gamma$.

In the first case we set $\hat{\gamma}_i = \gamma_i^{\max}, i = 1, 2$.

Let us analyze the second case in which we use the priority parameter q. Not all

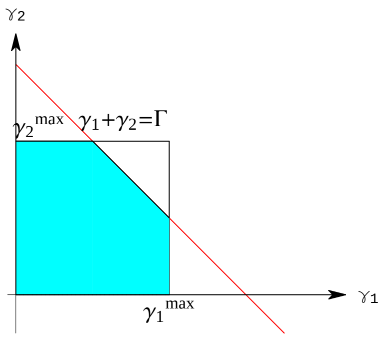

FIG. 4.1. *Case* $\Gamma_{in}^{\max} > \Gamma$.

packets can enter the junction, so let C be the amount of packets that can go through.

Then qC packets come from first incoming line and $(1-q)C$ packets from the second. Consider the space (γ_1, γ_2) and define the following lines:

$$r_q : \gamma_2 = \frac{1-q}{q}\gamma_1,$$

$$r_\Gamma : \gamma_1 + \gamma_2 = \Gamma.$$

Define P to be the point of intersection of the lines r_q and r_Γ. Recall that the final fluxes should belong to the region (see Figure 4.1):

$$\Omega = \{(\gamma_1, \gamma_2) : 0 \leq \gamma_i \leq \gamma_i^{\max}, i = 1, 2\}.$$

We distinguish two cases:

a) P belongs to Ω,

b) P is outside Ω.

In the first case we set $(\hat{\gamma}_1, \hat{\gamma}_2) = P$, while in the second case we set $(\hat{\gamma}_1, \hat{\gamma}_2) = Q$, with $Q = proj_{\Omega\cap r_\Gamma}(P)$ where $proj$ is the usual projection on a convex set, see Figure 4.2.

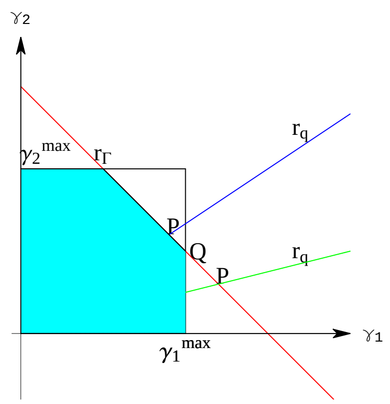

Fig. 4.2. *P belongs to Ω and P is outside Ω.*

The reasoning can be repeated also in the case of n incoming lines. In $\mathbb{R}^n$ the line r_q is given by $r_q = tv_q, t \in \mathbb{R}$, with $v_q \in \Delta_{n-1}$ where

$$\Delta_{n-1} = \left\{(\gamma_1, ..., \gamma_n) : \gamma_i \geq 0, i = 1, ..., n, \sum_{i=1}^{n} \gamma_i = 1\right\}$$

is the $(n-1)$ dimensional simplex and

$$H_\Gamma = \left\{(\gamma_1, ..., \gamma_n) : \sum_{i=1}^{n} \gamma_i = \Gamma\right\}$$

is a hyperplane where $\Gamma = \min\{\sum_{in} \gamma_i^{\max}, \sum_{out} \gamma_j^{\max}\}$. Since $v_q \in \Delta_{n-1}$, there exists a unique point $P = r_q \cap H_\Gamma$. If $P \in \Omega$, then we set $(\hat{\gamma}_1, ..., \hat{\gamma}_n) = P$. If $P \notin \Omega$, then we set $(\hat{\gamma}_1, ..., \hat{\gamma}_n) = Q = proj_{\Omega \cap H_\Gamma}(P)$, the projection over the subset $\Omega \cap H_\Gamma$. Observe that the projection is unique since $\Omega \cap H_\Gamma$ is a closed convex subset of H_Γ.

REMARK 4.6. *A possible alternative definition in the case $P \notin \Omega$ is to set $(\hat{\gamma}_1, ..., \hat{\gamma}_n)$ as one of the vertices of $\Omega \cap H_\Gamma$.*

As for the algorithm (RA1) $\hat{\pi}_i^{s,d} = \pi_{i,0}^{s,d}, i = 1, 2$.

Let us now determine $\hat{\gamma}_j, j = 3, 4$.

As for the incoming transmission lines we have to distinguish two cases :

I $\Gamma_{out}^{\max} = \Gamma$,

II $\Gamma_{out}^{\max} > \Gamma$.

In the first case $\hat{\gamma}_j = \gamma_j^{\max}, j = 3, 4$. Let us determine $\hat{\gamma}_j$ in the second case.

Recall α the traffic distribution parameter. Since not all packets can go on the outgoing transmission lines, we let C be the amount that goes through. Then αC packets go on the outgoing line I_3 and $(1-\alpha)C$ on the outgoing line I_4. Consider the space (γ_3, γ_4) and define the following lines:

$$r_\alpha : \gamma_4 = \frac{1-\alpha}{\alpha}\gamma_3,$$

$$r_\Gamma : \gamma_3 + \gamma_4 = \Gamma.$$

The line r_α can be computed from the matrix A. In fact, if we assume that a traffic distribution matrix A is assigned, then we compute $\hat{\gamma}_1, ...\hat{\gamma}_n$, and choose $v_\alpha \in \Delta_{m-1}$ by

$$v_\alpha = \Delta_{m-1} \cap \{tA(\hat{\gamma}_1, ...\hat{\gamma}_n) : t \in \mathbb{R}\},$$

where

$$\Delta_{m-1} = \left\{ (\gamma_{n+1}, ..., \gamma_{m+n}) : \gamma_{n+i} \geq 0, i = 1, ..., m, \sum_{i=1}^{n} \gamma_{n+i} = 1 \right\}$$

is the $(m-1)$ dimensional simplex.

We have to distinguish case 2a) and 2b) for the traffic distribution function.

Case 2a). Let us introduce the set

$$\mathcal{G} = \left\{ A\hat{\gamma}_{inc}^T : A \in \mathcal{A} \right\}.$$

LEMMA 4.7. *The set $\mathcal{G}$ is connected.*

Proof. The set $\mathcal{G}$ is the image of a connected set through a continuous map. Fixed $(\hat{\gamma}_1, \hat{\gamma}_2)$ the map is defined by

$$(\tilde{\alpha}_J^{1,s,d,3}, \tilde{\alpha}_J^{2,s,d,3}) \in [0,1] \times [0,1] \to (\Sigma, \hat{\gamma}_1 + \hat{\gamma}_2 - \Sigma),$$

where $\Sigma = \sum_{s,d} (\hat{\gamma}_1 \pi_1^{s,d} \tilde{\alpha}_J^{1,s,d,3} + \hat{\gamma}_2 \pi_2^{s,d} \tilde{\alpha}_J^{2,s,d,3})$. $\square$

Let us denote with G_1 and G_2 the endpoints of this set. Since in case 2a) we have an infinite number of matrices A, each of one determine a line r_α, we choose the most "natural" line r_α , i.e. the one nearest to the statistic line determined by measurements on the network.

Recall that the final fluxes should belong to the region:

$$\Omega = \left\{ (\gamma_3, \gamma_4) : 0 \leq \gamma_j \leq \gamma_j^{\max}, j = 3, 4 \right\}.$$

Define $P = r_\alpha \cap r_\Gamma, R = (\Gamma - \gamma_4^{\max}, \gamma_4^{\max}), Q = (\gamma_3^{\max}, \Gamma - \gamma_3^{\max})$. We distinguish 3 cases:

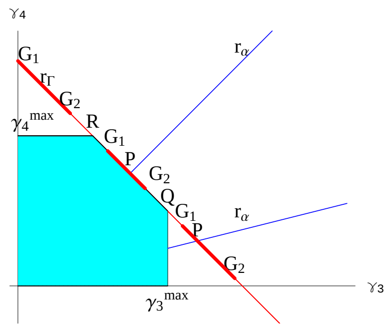

FIG. 4.3. *Traffic distribution function of type 2a)*

a) $\mathcal{G} \cap \Omega \cap r_\Gamma \neq \varnothing$,
b) $\mathcal{G} \cap \Omega \cap r_\Gamma = \varnothing$ and $\gamma_3(G_1) < \gamma_3(R)$,
c) $\mathcal{G} \cap \Omega \cap r_\Gamma = \varnothing$ and $\gamma_3(G_1) > \gamma_3^{\max}$.

If the set $\mathcal{G}$ has a priority over the line r_Γ we set $(\hat{\gamma}_3, \hat{\gamma}_4)$ in the following way. In case a) we define $(\hat{\gamma}_3, \hat{\gamma}_4) = proj_{\mathcal{G}\cap\Omega\cap r_\Gamma}(P)$, in case b) $(\hat{\gamma}_3, \hat{\gamma}_4) = R$, and finally in case c) $(\hat{\gamma}_3, \hat{\gamma}_4) = Q$.

Otherwise, if r_Γ has a priority over $\mathcal{G}$ we set $(\hat{\gamma}_3, \hat{\gamma}_4) = \min_{\gamma\in\Omega} \mathcal{F}(\gamma, r_\alpha, \mathcal{G})$ where $\mathcal{F}$ is a convex functional which depends on γ, r_α and on the set $\mathcal{G}$ of the routing standards. A possible choice of $\mathcal{F}$ is $\mathcal{F} = d(\gamma, B)$ where $B = w_1 r_\alpha + w_2 \int_{\mathcal{G}} r dr$ with w_1, w_2 real numbers and d denotes a distance.

The reasoning can be repeated also in the case of m outgoing lines.

The vector $\hat{\pi}_i^{s,d}, j = 3, 4$ are computed in the same way as for the algorithm (RA1).

Case 2b). In the case 2b) we have a unique matrix A and a unique vector v_α, so the fluxes on outgoing lines are computed as in the case without sources and destinations.

We distinguish two cases:
a) P belongs to Ω,
b) P is outside Ω.

In the first case we set $(\hat{\gamma}_3, \hat{\gamma}_4) = P$, while in the second case we set $(\hat{\gamma}_3, \hat{\gamma}_4) = Q$, where $Q = proj_{\Omega_{adm}}(P)$. Again, we can extend to the case of m outgoing lines as for the incoming lines defining the hyperplane $H_\Gamma = \{(\gamma_{n+1}, \ldots, \gamma_{n+m}) : \sum_{j=n+1}^{n+m} \gamma_j = \Gamma\}$ and choosing a vector $v_\alpha \in \Delta_{m-1}$.

Finally we define $\hat{\pi}_i^{s,d}, j = 3, 4$ as in the case 2a):

$$\hat{\pi}_j(t, a_j+, s, d) = \frac{\sum_{i=1}^{n} \alpha_J^{i,s,d,j} \pi_i^{s,d}(t, b_i-, s, d) f(\hat{\rho}_i)}{f(\hat{\rho}_j)}$$

for every $t \geq 0, j \in \{n+1, ..., n+m\}, s \in \mathcal{S}, d \in \mathcal{D}$.

REMARK 4.8. *Note that in case of algorithm (RA2) we find, separately, solution on incoming and outgoing lines.*

REMARK 4.9. *If* $\Gamma_{out}^{\max} < \Gamma_{in}^{\max}$ *we can define a different Riemann solver, considering a priority order of sending packets:* $(s_{k_1}, d_{l_1}) = c_1, (s_{k_2}, d_{l_2}) = c_2, (s_{k_3}, d_{l_3}) =$

$c_3, \ldots$*Packets are sent until the quantity of packets has been sent is equal to*

$$\sum_{\iota=1}^{\bar{\iota}} \sum_{i=1}^{n} \pi_i^{c_\iota} \gamma_{i0}$$

where $\bar{\iota}$ *is the minimum such that*

$$\sum_{\iota=1}^{\bar{\iota}} \sum_{i=1}^{n} \pi_i^{c_\iota} \gamma_{i0} > \Gamma.$$

Let us define $d = \Gamma - \sum_{\iota=1}^{\bar{\iota}} \sum_{i=1}^{n} \pi_i^{c_\iota} \gamma_{i0}$ *then*

$$\hat{\gamma}_1 = \sum_{\iota=1}^{\bar{\iota}-1} \pi_1^{c_\iota} \gamma_{10} + d/2,$$
$$\hat{\gamma}_2 = \sum_{\iota=1}^{\bar{\iota}-1} \pi_2^{c_\iota} \gamma_{20} + d/2.$$

Once solutions to Riemann problems are given, one can use a wave-front tracking algorithm to construct a sequence of approximate solutions. To pass to the limit one has to bound the number of waves and the BV norm of approximate solutions, see [5, 6]. In the next section we prove a BV bound on the density for the case of junctions with two incoming and two outgoing transmission lines, for both the routing algorithms.

5. Estimates on Density Variation. In this section we derive estimates on the total variation of the densities along a wave-front tracking approximate solution (constructed as in [6]) for the algorithm (RA2) with the traffic distribution function of type 2b). This allows to construct the solutions to the Cauchy problem in standard way, see [5].

Let us consider an admissible network $(N, \mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R})$. We assume that
(A1) every junction has at most two incoming and at most two outgoing lines.

This hypothesis is crucial, because the presence of more complicate junctions may provoke additional increases of the total variation of the flux and so of the density. The case where junctions have at most two incoming transmission lines and at most two outgoing ones can be treated in the same way.

From now on we fix a telecommunication network $(\mathcal{I}, \mathcal{J})$, with each node having at most two incoming and at most two outgoing lines, and a wave-front tracking approximate solution ρ, Π, defined on the telecommunication network.

Our aim is to prove an existence result for a solution (ρ, Π) in the case of a small perturbation of the equilibrium $(\bar{\rho}, \bar{\Pi})$. We have to analyze the following types of interactions:
I1. interaction of $\rho-$waves with $\rho-$waves on lines;
I2. interaction of $\rho-$waves with $\Pi-$waves on lines;
I3. interaction of $\Pi-$waves with $\Pi-$waves on lines;
I4. interaction of $\rho-$waves with junctions;
I5. interaction of $\Pi-$waves with junctions.

Observe that interaction of type I1 is classical and the total variation of the density decreases. Interaction of type I3 can not happen since $\Pi-$waves travel with speed depending only on the value of ρ.

5.1. Interaction of type I2. Let us consider a line I_i. We report some results proved in [11]. First we note that the characteristic speed of the density is smaller than the speed of a $\Pi-$wave, as follows from the lemma:

LEMMA 5.1. *Let $\rho \in [0,1]$ be a density and let $\lambda(\rho)$ be its characteristic speed. Then $\lambda(\rho) \leq v(\rho)$ and the equality holds if and only if $\rho = 0$.*

LEMMA 5.2. *Let us consider a shock wave connecting ρ^- and ρ^+. Then*

1. $\lambda(\rho^-, \rho^+) < v(\rho^-)$;
2. $\lambda(\rho^-, \rho^+) \leq v(\rho^+)$ *and the equality holds if and only if* $\rho^- = 0$.

LEMMA 5.3. *Let us consider a rarefaction shock fan connecting ρ^- and ρ^+. Then $v(\rho^+) > v(\rho^-) > f'(\rho^-)$.*

Putting together the previous lemmas we obtain the following result.

PROPOSITION 5.4. *An interaction of a ρ-wave with a Π-wave can happen only if the Π-wave interacts from the left respect to the ρ-wave. Moreover if this happens, then the ρ-wave does not change, while the Π-wave changes only its speed.*

5.2. Interaction of type I4. We consider interactions of ρ-waves with the junctions. In general these interactions produce an increment of the total variation of the flux and of the density in all the lines and a variation of the values of traffic-type functions on outgoing lines.

Fix a junction J with two incoming transmission lines I_1 and I_2 and two outgoing ones I_3 and I_4. Suppose that at some time $\bar{t}$ a wave interacts with the junction J and let $(\rho_1^-, \rho_2^-, \rho_3^-, \rho_4^-)$ and $(\rho_1^+, \rho_2^+, \rho_3^+, \rho_4^+)$ indicate the equilibrium configurations at the junction J before and after the interaction respectively. Introduce the following notation

$$\gamma_i^{\pm} = f(\rho_i^{\pm}), \quad \Gamma_{in}^{\pm} = \gamma_{1,\max}^{\pm} + \gamma_{2,\max}^{\pm}, \quad \Gamma_{out}^{\pm} = \gamma_{3,\max}^{\pm} + \gamma_{4,\max}^{\pm},$$

$$\Gamma^{\pm} = \min\{\Gamma_{in}^{\pm}, \Gamma_{out}^{\pm}\},$$

where $\gamma_{i,\max}^{\pm}, i = 1,2$ and $\gamma_{j,\max}^{\pm}, j = 3,4$ are defined as in (3.1) and (3.2). In general $-$ and $+$ denote the values before and after the interaction, while by Δ we indicate the variation, i.e. the value after the interaction minus the value before. For example $\Delta\Gamma = \Gamma^+ - \Gamma^-$. Let us denote by $TV(f)^{\pm} = TV(f(\rho(\bar{t}\pm, \cdot)))$ the flux variation of waves before and after the interaction, and

$$TV(f)_{in}^{\pm} = TV(f(\rho_1(\bar{t}\pm, \cdot))) + TV(f(\rho_2(\bar{t}\pm, \cdot))),$$

$$TV(f)_{out}^{\pm} = TV(f(\rho_3(\bar{t}\pm, \cdot))) + TV(f(\rho_4(\bar{t}\pm, \cdot))),$$

the flux variation of waves before and after the interaction, respectively, on incoming and outgoing lines.

Let us prove some estimates which are used later to control the total variation of the density function. For simplicity, from now on we assume that:

(A2) the wave interacting at time $\bar{t}$ with J comes from line 1 and we let ρ_1 be the value on the left of the wave.

The case of a wave from an outgoing line can be treated similarly.

LEMMA 5.5. *We have*

$$sgn\,(\Delta\gamma_3) \cdot sgn\,(\Delta\gamma_4) \geq 0.$$

LEMMA 5.6. *We have*

$$sgn(\gamma_1^+ - \gamma_1) \cdot sgn(\Delta\gamma_2) \geq 0,$$

where $\gamma_1 = f(\rho_1)$.

LEMMA 5.7. *It holds*

$$TV(f)^+_{out} = |\Delta\Gamma|.$$

LEMMA 5.8. *We have*

$$TV(f)^-_{in} = TV(f)^+_{in} + |\Delta\Gamma|. \tag{5.1}$$

From Lemmas 5.7 and 5.8, we are ready to state the following:

LEMMA 5.9. *It holds*

$$TV(f)^+ = CTV(f)^-.$$

A ρ-wave produces a Π-wave, but the following lemma holds:

LEMMA 5.10. *Let J be a junction with at most two incoming lines and two outgoing ones. Suppose that a ρ-wave (ρ_1, ρ_{10}) approaches the junction J. If there exists $\delta > 0$ such that $f(\rho_1) > \delta > 0$, $f(\rho_{1,0}) > \delta > 0$ then there exists $C > 0$, such that the variation of the traffic-type functions in outgoing lines is bounded by C times the flux variation of the interacting wave, i.e.*

$$TV(\Pi)^+ \leq \frac{C}{\delta} TV(f)^-.$$

Proof. Fix a source $s \in \mathcal{S}$ and a destination $d \in \mathcal{D}$. We denote by $\pi_{i,0}, \rho_{i,0}$ and $\hat{\pi}_i, \hat{\rho}_i (i \in \{1,2,3,4\})$ the values of the densities and of the traffic-type functions for s and d at J, respectively, before and after the interaction of the $\rho-$wave with J. We have for $j \in \{3,4\}$

$$|\pi^{s,d}_{j,0} - \hat{\pi}^{s,d}_j| =$$

$$\left| \frac{\alpha_J^{1,s,d,j} \pi^{s,d}_{1,0} f(\rho_{1,0})}{f(\rho_{j,0})} + \frac{\alpha_J^{2,s,d,j} \pi^{s,d}_{2,0} f(\rho_{2,0})}{f(\rho_{j,0})} - \frac{\alpha_J^{1,s,d,j} \pi^{s,d}_{2,0} f(\hat{\rho}_1)}{f(\hat{\rho}_j)} - \frac{\alpha_J^{2,s,d,j} \pi^{s,d}_{2,0} f(\hat{\rho}_2)}{f(\hat{\rho}_j)} \right| \leq$$

$$\frac{\alpha_J^{1,s,d,j} \pi^{s,d}_{1,0}}{f(\rho_{j,0}) f(\hat{\rho}_j)} |f(\rho_{1,0}) f(\hat{\rho}_j) - f(\hat{\rho}_1) f(\rho_{j,0})| + \frac{\alpha_J^{2,s,d,j} \pi^{s,d}_{2,0}}{f(\rho_{j,0}) f(\hat{\rho}_j)} |f(\rho_{2,0}) f(\hat{\rho}_j) - f(\hat{\rho}_2) f(\rho_{j,0})| \leq$$

$$\frac{C'}{\delta^2} |f(\rho_{1,0})(f(\hat{\rho}_j) - f(\rho_{j,0})) + f(\rho_{j,0})(f(\rho_{1,0}) - f(\hat{\rho}_1))| +$$

$$\frac{C'}{\delta^2} |f(\rho_{2,0})(f(\hat{\rho}_j) - f(\rho_{j,0})) + f(\rho_{j,0})(f(\rho_{2,0}) - f(\hat{\rho}_2))| \leq$$

$$\frac{C'}{\delta^2} f(\rho_{1,0}) |f(\hat{\rho}_j) - f(\rho_{j,0})| + \frac{C'}{\delta^2} f(\rho_{j,0}) |f(\rho_{1,0}) - f(\hat{\rho}_1)| +$$

$$\frac{C'}{\delta^2} f(\rho_{2,0})|f(\hat{\rho}_j) - f(\rho_{j,0})| + \frac{C'}{\delta^2} f(\rho_{j,0})|f(\rho_{2,0}) - f(\hat{\rho}_2)| =$$

$$\frac{C'}{\delta}|f(\hat{\rho}_j) - f(\rho_{j,0})| + \frac{C'}{\delta}|f(\rho_{1,0}) - f(\hat{\rho}_1)| +$$

$$\frac{C'}{\delta}|f(\hat{\rho}_j) - f(\rho_{j,0})| + \frac{C'}{\delta}|f(\rho_{2,0}) - f(\hat{\rho}_2)| =$$

$$\frac{C'}{\delta}(|f(\hat{\rho}_j) - f(\rho_{j,0})| + |f(\rho_{1,0}) - f(\hat{\rho}_1)| + |f(\rho_{2,0}) - f(\hat{\rho}_2)|) \leq 2\frac{C'}{\delta} TV(f)^-$$

with a suitable constant C'. Set $C = 2C'$. □

5.3. Interaction of type I5. We consider interactions of Π-waves with the junctions. Since Π-waves have always positive speed, they can interact with the junction only from an incoming line.

LEMMA 5.11. *Let us consider a junction J and a Π-wave on an incoming line I_i interacting with J. If A is the distributional matrix for J, whose entries are given by (3.3), then the interaction of the Π-wave with J modifies only the i-th column of A. Moreover the variation of the i-th column is bounded by the Π-wave variation.*

Proof. For each $s \in \mathcal{S}$ and a destination $d \in \mathcal{D}$, we denote by $\pi_i^{s,d}$ and $\pi_{i,0}^{s,d}$, respectively, the left and the right states of the Π-wave. Moreover, for every $j \in \{3, 4\}$, we denote with $\alpha_{j,i}^-$ and $\alpha_{j,i}^+$, respectively, the entries of the matrix A before and after the interaction of the Π-wave with J. By (3.3), it is clear that, if $l \neq i$, then the entries $\alpha_{j,l}$ are not modified. For $l = i$, we have

$$|\alpha_{j,i}^+ - \alpha_{j,i}^-| = \left| \sum_{s\in\mathcal{S},d\in\mathcal{D}} \pi_i^{s,d} \alpha_J^{i,s,d,j} - \sum_{s\in\mathcal{S},d\in\mathcal{D}} \pi_{i,0}^{s,d} \alpha_J^{i,s,d,j} \right| \leq$$

$$\sum_{s\in\mathcal{S},d\in\mathcal{D}} |\pi_i^{s,d} - \pi_{i,0}^{s,d}| \alpha_J^{i,s,d,j}.$$

This completes the proof. □

LEMMA 5.12. *Let us consider a junction J and a Π-wave on an incoming line I_i interacting with J. Then there exists $C > 0$, such that the variation of the fluxes is bounded by C times the Π-wave variation, i.e.*

$$TV(f)^+ \leq CTV(\Pi)^-.$$

Proof. For simplicity let us consider the case $P \in \Omega$ where

$$\Omega = \left\{(\gamma_1, \gamma_2) \in \Omega_1 \times \Omega_2 : A(\gamma_1, \gamma_2)^T \in \Omega_3 \times \Omega_4\right\}.$$

Since the solution of the Riemann Problem depends on the position of the traffic distribution line r_α we consider

$$|A(\pi)\gamma_{inc}^T - A(\hat{\pi})\gamma_{inc}^T| = |(A(\pi) - A(\hat{\pi}))\gamma_{inc}^T| =$$

$$\left| \begin{pmatrix} \alpha_{3,1}(\pi) - \alpha_{3,1}(\hat{\pi}) & \alpha_{3,2}(\pi) - \alpha_{3,2}(\hat{\pi}) \\ \alpha_{4,1}(\pi) - \alpha_{4,1}(\hat{\pi}) & \alpha_{4,2}(\pi) - \alpha_{4,2}(\hat{\pi}) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right| =$$

$$|(\alpha_{3,1}(\pi)-\alpha_{3,1}(\hat{\pi}))\gamma_1+(\alpha_{3,2}(\pi)-\alpha_{3,2}(\hat{\pi}))\gamma_2,(\alpha_{4,1}(\pi)-\alpha_{4,1}(\hat{\pi}))\gamma_1+(\alpha_{4,2}(\pi)-\alpha_{4,2}(\hat{\pi}))\gamma_2| =$$

$$|(\alpha_{3,1}(\pi)-\alpha_{3,1}(\hat{\pi}),\alpha_{4,1}(\pi)-\alpha_{4,1}(\hat{\pi}))\gamma_1+(\alpha_{3,2}(\pi)-\alpha_{3,2}(\hat{\pi}),\alpha_{4,2}(\pi)-\alpha_{4,2}(\hat{\pi}))\gamma_2| \le$$

$$\gamma_1|(\alpha_{3,1}(\pi)-\alpha_{3,1}(\hat{\pi}),\alpha_{4,1}(\pi)-\alpha_{4,1}(\hat{\pi}))|+\gamma_2|(\alpha_{3,2}(\pi)-\alpha_{3,2}(\hat{\pi}),\alpha_{4,2}(\pi)-\alpha_{4,2}(\hat{\pi}))| =$$

$$\gamma_1\left|\left(\sum_{s\in\mathcal{S},d\in\mathcal{D}}(\pi_{1,0}^{s,d}-\hat{\pi}_1^{s,d})\alpha_J^{1,s,d,3},\sum_{s\in\mathcal{S},d\in\mathcal{D}}(\pi_{1,0}^{s,d}-\hat{\pi}_1^{s,d})\alpha_J^{1,s,d,4}\right)\right|+$$

$$\gamma_2\left|\left(\sum_{s\in\mathcal{S},d\in\mathcal{D}}(\pi_{2,0}^{s,d}-\hat{\pi}_2^{s,d})\alpha_J^{2,s,d,3},\sum_{s\in\mathcal{S},d\in\mathcal{D}}(\pi_{2,0}^{s,d}-\hat{\pi}_2^{s,d})\alpha_J^{2,s,d,4}\right)\right| =$$

$$\gamma_1\left|\sum_{s\in\mathcal{S},d\in\mathcal{D}}\left((\pi_{1,0}^{s,d}-\hat{\pi}_1^{s,d})\alpha_J^{1,s,d,3},(\pi_{1,0}^{s,d}-\hat{\pi}_1^{s,d})\alpha_J^{1,s,d,4}\right)\right|+$$

$$\gamma_2\left|\sum_{s\in\mathcal{S},d\in\mathcal{D}}\left((\pi_{2,0}^{s,d}-\hat{\pi}_2^{s,d})\alpha_J^{2,s,d,3},(\pi_{2,0}^{s,d}-\hat{\pi}_2^{s,d})\alpha_J^{2,s,d,4}\right)\right| =$$

$$\gamma_1\left|\sum_{s\in\mathcal{S},d\in\mathcal{D}}(\pi_{1,0}^{s,d}-\hat{\pi}_1^{s,d})(\alpha_J^{1,s,d,3},\alpha_J^{1,s,d,4})\right|+$$

$$\gamma_2\left|\sum_{s\in\mathcal{S},d\in\mathcal{D}}(\pi_{2,0}^{s,d}-\hat{\pi}_2^{s,d})(\alpha_J^{2,s,d,3},\alpha_J^{2,s,d,4})\right| \le$$

$$\gamma_1\sum_{s\in\mathcal{S},d\in\mathcal{D}}|\pi_{1,0}^{s,d}-\hat{\pi}_1^{s,d}||(\alpha_J^{1,s,d,3},\alpha_J^{1,s,d,4})|+\gamma_2\sum_{s\in\mathcal{S},d\in\mathcal{D}}|\pi_{2,0}^{s,d}-\hat{\pi}_2^{s,d}||(\alpha_J^{2,s,d,3},\alpha_J^{2,s,d,4})| =$$

$$\sum_{s\in\mathcal{S},d\in\mathcal{D}}(\gamma_1|\pi_{1,0}^{s,d}-\hat{\pi}_1^{s,d}||(\alpha_J^{1,s,d,3},\alpha_J^{1,s,d,4})|+\gamma_2|\pi_{2,0}^{s,d}-\hat{\pi}_2^{s,d}||(\alpha_J^{2,s,d,3},\alpha_J^{2,s,d,4})|) \le$$

$$C\sum_{s\in\mathcal{S},d\in\mathcal{D}}(|\pi_{1,0}^{s,d}-\hat{\pi}_1^{s,d}|+|\pi_{2,0}^{s,d}-\hat{\pi}_2^{s,d}|)$$

for some constant C. □

5.4. Existence of solutions for equilibria perturbations. Let us consider an admissible network $(N, \mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R})$. We have the following theorem.

PROPOSITION 5.13. *Let $(\bar{\rho}, \bar{\Pi})$ be an equilibrium on the whole network such that $f(\bar{\rho}) > \delta > 0$. Define $\hat{\lambda} = \max\{f'(0), -f'(1)\}$ and*

$$\Delta t = \frac{\min_i(b_i - a_i)}{\hat{\lambda}},$$

which represents the minimum time for a wave to go from a junction to another one. For $0 < \varepsilon < \delta/\hat{\lambda}$ *there exists* $\tilde{t} = \tilde{t}(\delta, \varepsilon)$ *such that the following holds. For every perturbation* $(\tilde{\rho}, \tilde{\Pi})$ *of the equilibrium with*

$$\|\tilde{\rho}\|_{BV} \le \varepsilon, \left\|\tilde{\Pi}\right\|_{BV} \le \varepsilon$$

and

$$\|\tilde{\rho} - \bar{\rho}\|_{\infty} \le \varepsilon, \left\|\tilde{\Pi} - \bar{\Pi}\right\|_{\infty} \le \varepsilon$$

there exists an admissible solution (ρ, Π) *defined for every* $t \in [0, \tilde{t}]$ *with initial datum* $(\tilde{\rho}, \tilde{\Pi})$.

Proof. Denote with (ρ_ν, Π_ν) a sequence of approximate wave-front tracking solutions with initial data approximating $(\tilde{\rho}, \tilde{\Pi})$. Let us introduce the following notations:

$$TV(f(\rho_\nu(k\Delta t, \cdot))) = Tf_k,$$

$$TV(\Pi_\nu(k\Delta t, \cdot)) = T\Pi_k.$$

For every interaction of a wave with a junction we have the estimates of Lemmas 5.9, 5.10 and 5.12, therefore

$$Tf_k \le Tf_{k-1} + CT\Pi_{k-1},$$

$$T\Pi_k \le T\Pi_{k-1} + \frac{C}{\delta_k} Tf_{k-1},$$

where $\delta_k = \delta_{k-1} - TVf_{k-1}$ and δ_0 is such that $f(\tilde{\rho}) > \delta_0 > 0$ (notice that $\delta_0 > \delta - \hat{\lambda}\epsilon$.) Setting

$$T_k = \max_k (Tf_k, T\Pi_k),$$
$$\delta_k = \delta_{k-1} - T_k,$$

we obtain

$$T_k \le \left(\frac{C}{\delta_k} + 1\right) T_{k-1}.$$

The exact computation of the not explosion time for the variation is a bit involved, so let us assume, for simplicity, that δ is small and consider a continuous evolution. Defining $\delta(t) = \delta_0 - T(t)$ we obtain

$$\dot{T}(t) \le \frac{C}{\delta} T(t) = \frac{CT(t)}{\delta_0 - T(t)}$$

from which we get $\delta_0 \ln(T) - T = \delta_0 \ln(T_0) + CT - T_0$, that implicitly define $T = T(t, \delta_0, T_0)$. Define $\hat{t}$ such that $T(\hat{t}, \delta_0, T_0) = +\infty$, then for $t \le \tilde{t} = \hat{t}/2$ there exists a constant $C_1 > 0$ such that

$$TV(f(\rho_\nu(t, \cdot))) \le C_1$$

$$TV(\Pi_\nu(t,\cdot)) \leq C_1,$$

uniformly in ν.
Now, by Helly theorem, Π_ν and $f(\rho_\nu)$ converge by subsequences strongly in L^1. Moreover, again by subsequences, ρ_ν converges weakly in L^1_{loc}. We then can complete the proof as in [6]. □

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