PACKETS FLOW ON TELECOMMUNICATION NETWORKS

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Abstract. The aim of this paper is to introduce a macroscopic fluid dynamic model dealing with the flows of information on a telecommunication network encoded in packets. Taking an intermediate time and space scale, we propose a model similar to that introduced recently for car traffic, see [11]. For dynamics at nodes we consider two "routing algorithms" and prove existence of solutions to Cauchy problems. The main difference among the two algorithms is the possibility of redirecting packets of the second, which in turn implies stability, i.e. Lipschitz continuous dependence from initial data, not granted for solutions using the first algorithm.

1. Introduction. The aim of this paper is to introduce a macroscopic fluid dynamic model dealing with the flows of information on a telecommunication network encoded in packets. There are some recent works on traffic flow on road networks, see [10, 11, 14, 15, 16, 18], that are based on macroscopic description via car densities and other conserved quantities. Our idea is to look at the network at an intermediate time scale so that packets transmission happens at a faster level but the equilibria of the whole network are reached only as asymptotic. This permits to construct a model again relying on macroscopic description.

There exist various approaches to traffic flow on telecommunication networks, in particular for Internet and with special focus on properties of control congestion algorithms as TCP/IP, see for example [4, 17, 24]. Our idea is rather to take a large number of nodes, which use some simple routing algorithm, and via some limiting procedure obtain a partial differential equation for the packet density on the network. First we focus on a straight transmission line and justify the limiting procedure. Then we pass to consider a network and introduce two routing algorithms for nodes with many entering and exiting lines. Let us start from the basic assumptions.

A network is formed by a finite collection of transmission lines and nodes (or routers). We assume that each node receives and sends information encoded in packets. Each packet can thus be seen as a particle on the network, but we have to take into account specific issues of telecommunications. Having in mind Internet as key model, it is assumed that:

1) Each packet travels on the network with a fixed speed and with assigned final destination;
2) Nodes receive, process and then forward packets. Packets may be lost with a probability increasing with the number of packets to be processed. Each lost packet is sent again.

We first model the behavior of a single straight transmission line on which there are some consecutive nodes. Each node sends packets to the following one a first time, then packets which are lost in this process are sent a second time and so on. The important point is that each packet is sent until it reaches next node, thus, looking at macroscopic level, it is assumed that packets are conserved. This leads for the microscopic dynamics to the simple model consisting of a single conservation law:

$$\rho_t + f(\rho)_x = 0, \tag{1.1}$$

where ρ is the packet density, v is the velocity and $f(\rho) = v\rho$ is the flux. Since the packet transmission velocity on the line is assumed constant, we can derive an average transmission velocity among nodes considering the amount of packets that may be lost. More precisely, assigning a loss probability as function of the density, it is possible to compute a velocity function and thus a flux function.

The conclusion is rigorously justified only for constant density, but is assumed to hold in general. This corresponds to the hypotheses that macroscopic density waves move at a velocity much smaller than the packets transmission velocity. In Section 2 we derive some models and then we focus the rest of the paper on a particular one that implies equivalence between the total variation of density and of flux. Even if our limiting procedure is not completely rigorous, there are other approaches, as [3] for supply chains, which lead to conservation laws. Moreover, since our method to solve problems at nodes is based only on flux values, every limiting procedure, which leads to a conservation law formulation, may be used to treat the problem on a network.

The aim is then to consider complex networks, thus we need to introduce a way of solving dynamics at nodes in which many lines intersect. For this, respecting rule 2), we propose two different routing algorithms:

(RA1) Packets from incoming lines are sent to outgoing ones according to their final destination (without taking into account possible high loads of outgoing lines).

(RA2) Packets are sent to outgoing lines in order to maximize the flux through the node.

The main differences of the two algorithms are the following. The first one simply send each packet to the outgoing line which is naturally chosen according to the final destination of the packet itself. The algorithm is blind to possible overloads of some outgoing lines and, by some abuse of notation, is similar to the behavior of a "switch". The second algorithm, on the contrary, send packets to outgoing lines taking into account the loads, and thus possibly redirecting packets. Again by some abuse of notation, this is similar to a "router" behavior.

The routing algorithm (RA1) can be described by two rules and was already used in [11] for car traffic. In particular a traffic distribution matrix A is given, which describes the percentage of packets from an incoming line that are addressed to an outgoing one. For existence of solutions to the Cauchy problem on the network, we have to restrict to the case of simple nodes with two incoming and two outgoing lines, but, differently from [11], we can obtain a precise bound on the total variation of density, thanks to the assumption on the flux function, and then derive existence of solutions to Cauchy problem more directly by wave-front tracking. However, Lipschitz continuous dependence of solutions is not granted.

Then we pass to analyze the routing algorithm (RA2). Notice that this second algorithm was not considered for car traffic, because redirection of cars is not expected from modelling point of view (except special situations as closure of a road).

In order to determine unique solutions to Riemann problems, some additional parameters are introduced, called respectively priority parameters and traffic distribution parameters. The former describe priorities among incoming lines, while the latter have the same meaning of the traffic distribution matrix.

The advantage of this second algorithm is that the flux variation at a node is conserved for interaction of waves from transmission lines. This permits us both to obtain estimates on the total variation of density, thus to construct solutions again by wave-front tracking, and also to obtain uniqueness and Lipschitz continuous dependence of solutions. The latter result is achieved by the method introduced in [6, 8], which considers a Riemannian type metric on L^1. More precisely, the distance among solutions is measured by paths in L^1 which admit some generalized tangent vectors. The key point is that the norms of tangent vectors are known to decrease inside each line (i.e. for scalar conservation laws), while for interactions with nodes its evolution is determined by flux variation. As explained in Section 5.2.1, other known methods, to treat uniqueness for scalar conservation laws, seems not to work for the network case.

The obtained results show the strong effect of the routing algorithm. More precisely, the choice of a "router" type algorithm, i.e. (RA2), implies stability of solutions, with respect of perturbation of the data, opposed to the instability obtained with the "switch" type ones.

The paper is organized as follows. Section 2 describes the dynamics of packet density on a single transmission line. Section 3 gives general definition of network and of Riemann solver. Then, we describe the two routing algorithms in Section 4, giving explicit unique solutions to Riemann problems. Finally, Section 5 provides the needed estimates for constructing solutions to Cauchy problems and to obtain continuous dependence for the second algorithm.

2. Packets loss and velocity functions on transmission lines. We model a transmission line by a sequence of nodes N_k, representing routers, and edges which connect consecutive nodes. Thus the transmission line is represented by a real interval I union of many edges and nodes.

Each node (router) sends and receives packets. Following rule 1), we assume that packets flow at constant velocity from each node N_k to N_{k+1}. Taking a discrete time scale for the evolution, the state at time t_i is described by the packets quantities $R_k(t_i)$ on nodes N_k and transmission happens among consecutive nodes between two discrete times. Therefore, to determine the dynamics on I we need to describe the effect of packets loss on the velocity of transmission function.

As for the Internet, we assume that each node N_k sends again packets that are lost by the following node N_{k+1}. Therefore the number of packets is conserved, i.e. at macroscopic level we expect (1.1) to hold. More precisely, we assume that there exists a function $p : [0, R_{max}] \to [0, 1]$ which assigns the packet loss probability as function of the number of packets.

Let us focus now on two consecutive nodes and introduce some notation. Suppose that δ is the distance between the nodes N_k and N_{k+1}. Let Δt_0 be the transmission time of packets from node N_k to node N_{k+1} if they are sent with success at the first attempt, and Δt_{av} the average transmission time when some packets are lost by N_{k+1}. Finally, we denote with $\bar{v} = \frac{\delta}{\Delta t_0}$ and $v = \frac{\delta}{\Delta t_{av}}$ the packets velocity in the two cases.

At the first attempt, the packets sent by node N_k reach with success node N_{k+1} with probability $(1 - p)$ and they are lost by node N_{k+1} with probability p. At the second attempt there are p of the total number of packets left to be sent again and $(1 - p)p$ are sent with success while p^2 are lost. Going on at the n-th attempt $(1 - p)p^{n-1}$ packets are sent successfully and p^n are lost. The average transmission time is equal to

$$\Delta t_{av} = \sum_{n=1}^{+\infty} n\Delta t_0 (1 - p)p^{n-1} = \frac{\Delta t_0}{1 - p}, \tag{2.1}$$

from which we get that the transmission velocity is given by

$$v = \frac{\delta}{\Delta t_{av}} = \frac{\delta}{\Delta t_0}(1 - p) = \bar{v}(1 - p). \tag{2.2}$$

The above reasoning works for the entire line if $R_k(t_0) = R$ for all k. In fact, one gets immediately that $R_k(t_i) = R$ for all i and k thus it holds:

LEMMA 2.1. *Assume that $R_k(t_0) = R$ for all k. Then the average transmission time and velocity are given by (2.1) and (2.2).*

Clearly Lemma 2.1 gives an average velocity only if the density is constant. However, we assume the conclusion to hold in general for the macroscopic velocity and use this together with equation (1.1). This assumption is not completely justified but it is reasonable if the transmission velocity of packets is expected to be much bigger than the macroscopic velocity.

We may also assign the loss probability directly as function of the packet density, then the corresponding flux is easily determined. Such loss probability should vanish for low load levels of nodes and reach the value 1 for $R = R_{max}$. We show some choice of packets loss functions and the corresponding macroscopic fluxes.

EXAMPLE 2.2. *Let us suppose that the packets loss probability is given by*

$$p(\rho) = \begin{cases} 0, & 0 \le \rho \le \sigma, \\ \frac{2(\rho - \sigma)}{\rho}, & \sigma \le \rho \le \rho_{\max}, \end{cases}$$

for some $\sigma \in]0, \rho_{max}[$. Then the average transmission velocity is equal to

$$v(\rho) = \bar{v}(1 - p(\rho)) = \begin{cases} \bar{v}, & 0 \le \rho \le \sigma, \\ \bar{v}\frac{(2\sigma - \rho)}{\rho}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

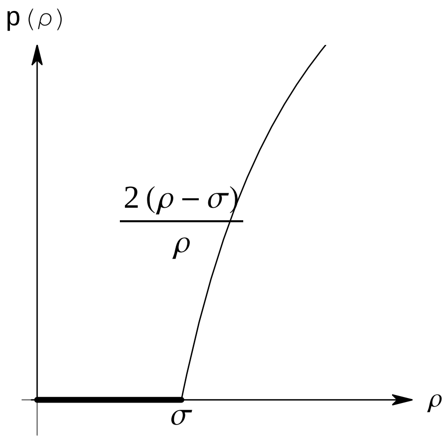

FIG. 2.1. *Packets loss function.*

Imposing that

$$v(\rho_{\max}) = \bar{v}\frac{(2\sigma - \rho_{\max})}{\rho_{\max}} = 0,$$

we get that $\sigma = \frac{\rho_{\max}}{2}$*. Since* $f(\rho) = v(\rho)\rho$ *it follows that*

$$f(\rho) = \begin{cases} \bar{v}\rho, & 0 \leq \rho \leq \sigma, \\ \bar{v}(2\sigma - \rho), & \sigma \leq \rho \leq \rho_{\max}. \end{cases}$$

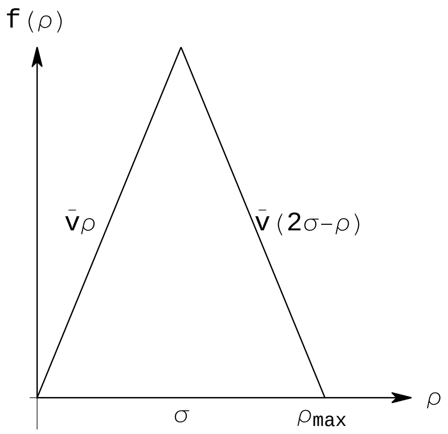

FIG. 2.2. *Flux function.*

The fundamental diagram (i.e. the expression of the flux as function of the density) of Example 2.2 was extensively used in traffic flow literature, see [13, 20], and is sometimes called the Daganzo-Newell flux.

EXAMPLE 2.3. *Suppose that*

$$p(\rho) = \begin{cases} 0, & 0 \leq \rho \leq \sigma, \\ \frac{\rho - \sigma}{\sigma}, & \sigma \leq \rho \leq \rho_{\max}. \end{cases}$$

It follows that

$$v(\rho) = \begin{cases} \bar{v}, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}(2\sigma-\rho)}{\sigma}, & \sigma \le \rho \le \rho_{\max}, \end{cases}$$

and

$$f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}\rho(2\sigma-\rho)}{\sigma}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

EXAMPLE 2.4. *Suppose that*

$$p(\rho) = \begin{cases} 0, & 0 \le \rho \le \sigma, \\ \frac{(\rho-\sigma)^2}{\sigma^2}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

It follows that

$$v(\rho) = \begin{cases} \bar{v}, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}\rho(2\sigma-\rho)}{\sigma^2}, & \sigma \le \rho \le \rho_{\max}, \end{cases}$$

and

$$f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}\rho^2(2\sigma-\rho)}{\sigma^2}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

REMARK 2.5. *Examples 2.2 and 2.3 lead to fluxes which are not* $\mathcal{C}^1$*, the opposite happens for Example 2.4. Notice that only for Example 2.2 the corresponding flux has the property that* $f'(\rho\pm) \neq 0$ *for every* ρ*. Thus the density variation along discontinuities not crossing* σ *is equivalent to the flux ones.*

In what follows we suppose that measures on packets loss probability lead to the formulation of Example 2.2. This allows to control the variation of the density function in terms of the variation of the flux function as shown later.

We can suppose for simplicity that $\rho_{\max} = 1$, so we have the following assumptions on the flux:

(F) $f : [0,1] \to R$, $\quad f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \bar{v}(2\sigma-\rho), & \sigma \le \rho \le 1, \end{cases}$

$f(0) = f(1) = 0$. Thus $\sigma = \frac{1}{2}$ is the unique maximum point.

3. Telecommunication networks. We consider a telecommunication network, that is modelled by a finite set of intervals $I_i = [a_i, b_i] \subset \mathbb{R}, i = 1, ..., N, a_i < b_i$, possibly with either $a_i = -\infty$ or $b_i = +\infty$, on which we consider the model of the previous section, i.e. equation (1.1) with assumption (F). The network evolution is described by a finite set of functions ρ_i defined on $[0, +\infty[\times I_i$.

On each transmission line I_i we want ρ_i to be a weak entropic solution of (1.1), that is for every function $\varphi : [0, +\infty[\times I_i \to \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left(\rho_i \frac{\partial \varphi}{\partial t} + f(\rho_i) \frac{\partial \varphi}{\partial x} \right) dxdt = 0, \tag{3.1}$$

and for every $k \in \mathbb{R}$ and every $\tilde{\varphi} : [0, +\infty[\times I_i \to \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left(|\rho_i - k| \frac{\partial \tilde{\varphi}}{\partial t} + sgn(\rho_i - k) (f(\rho_i) - f(k)) \frac{\partial \tilde{\varphi}}{\partial x} \right) dxdt \ge 0. \tag{3.2}$$

It is well known that, for equation (1.1) on $\mathbb{R}$ and for every initial data in L^∞, there exists a unique weak entropic solution depending in a continuous way from the initial data in L^1_{loc}. Moreover, for initial data in $L^\infty \cap L^1$ we have Lipschitz continuous dependence in L^1.

We assume that the transmission lines are connected by some junctions. Each junction J is given by a finite number of incoming transmission lines and a finite number of outgoing transmission lines, thus we identify J with $((i_1, ..., i_n), (j_1, ..., j_m))$ where the first n-tuple indicates the set of incoming transmission lines and the second m-tuple indicates the set of outgoing transmission lines. Each transmission line can be incoming transmission line for at most one junction and outgoing for at most one junction. Hence the complete model is given by a couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I} = \{I_i : i = 1, ..., N\}$ is the collection of transmission lines and $\mathcal{J}$ is the collection of junctions.

Now we discuss how to define solutions at junctions. For this, fix a junction J with n incoming transmission lines, say $I_1, ..., I_n$, and m outgoing transmission lines, say $I_{n+1}, ..., I_{n+m}$. A weak solution at J is a collection of functions $\rho_l : [0, +\infty[\times I_l \to \mathbb{R}, l = 1, ..., n+m$, such that

$$\sum_{l=1}^{n+m} \left(\int_0^{+\infty} \int_{a_l}^{b_l} \left(\rho_l \frac{\partial \varphi_l}{\partial t} + f(\rho_l) \frac{\partial \varphi_l}{\partial x} \right) dx dt \right) = 0, \tag{3.3}$$

for every $\varphi_l, l = 1, ..., n+m$, smooth having compact support in $]0, +\infty[\times]a_l, b_l]$ for $l = 1, ..., n$ (incoming transmission lines) and in $]0, +\infty[\times [a_l, b_l[$ for $l = n+1, ..., n+m$ (outgoing transmission lines), that are also smooth across the junction, i.e.

$$\varphi_i(\cdot, b_i) = \varphi_j(\cdot, a_j), \quad \frac{\partial \varphi_i}{\partial x}(\cdot, b_i) = \frac{\partial \varphi_j}{\partial x}(\cdot, a_j), \quad i = 1, ..., n, j = n+1, ..., n+m.$$

Remark 3.1. *Let $\rho = (\rho_1, ..., \rho_{n+m})$ be a weak solution at the junction such that each $x \to \rho_i(t, x)$ has bounded variation. We can deduce that ρ satisfies the Rankine-Hugoniot condition at the junction J, namely*

$$\sum_{i=1}^{n} f(\rho_i(t, b_i-)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, a_j+)), \tag{3.4}$$

for almost every $t > 0$.

For a scalar conservation law a Riemann problem is a Cauchy problem for an initial data of Heavyside type, that is piecewise constant with only one discontinuity. One looks for centered solutions, i.e. $\rho(t, x) = \phi(\frac{x}{t})$ formed by simple waves, which are the building blocks to construct solutions to the Cauchy problem via wave- front tracking algorithm. These solutions are formed by continuous waves called rarefactions and by travelling discontinuities called shocks. The speed of waves are related to the values of f', see [7].

Analogously, we call Riemann problem for a junction the Cauchy problem corresponding to an initial data which is constant on each transmission line.

Definition 3.2. *A Riemann Solver for the junction J is a map $RS : [0,1]^n \times [0,1]^m \to [0,1]^n \times [0,1]^m$ that associates to Riemann data $\rho_0 = (\rho_{1,0}, \ldots, \rho_{n+m,0})$ at J a vector $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_{n+m})$ so that the solution on an incoming transmission line $I_i, i = 1, ..., n$, is given by the wave $(\rho_{i,0}, \hat{\rho}_i)$ and on an outgoing one $I_j, j = n+1, ..., n+m$, is given by the wave $(\hat{\rho}_j, \rho_{j,0})$. We require the consistency condition*

(CC) $RS(RS(\rho_0)) = RS(\rho_0)$.

Remark 3.3. *The condition (CC) is necessary for a good definition of Riemann solver and thus also for uniqueness.*

Assume for example that $RS(\rho) = \rho'$ and $RS(\rho') = \rho$ for some Riemann data $\rho \neq \rho'$. To solve the Riemann problem with datum ρ, one should use the boundary datum ρ' at the junction.

In turn, when ρ' starts propagating into lines, one should go back to ρ and so on and so forth. A solution would thus not exist.

The same kind of problem happens for uniqueness.

Once a Riemann solver is assigned we can define admissible solutions at J.

DEFINITION 3.4. *Assume a Riemann Solver RS is assigned. Let $\rho = (\rho_1, ..., \rho_{n+m})$ be such that $\rho_i(t, \cdot)$ is of bounded variation for every $t \geq 0$. Then ρ is an admissible weak solution of (1.1) related to RS at the junction J if and only if the following properties hold:*

(i) ρ is a weak solution at the junction J;

(ii) for almost every t setting

$$\rho_J(t) = (\rho_1(\cdot, b_1-), \ldots, \rho_n(\cdot, b_n-), \rho_{n+1}(\cdot, a_{n+1}+), \ldots, \rho_{n+m}(\cdot, a_{n+m}+))$$

we have

$$RS(\rho_J(t)) = \rho_J(t).$$

For every transmission line $I_i = [a_i, b_i]$, if $a_i > -\infty$ and I_i is not the outgoing transmission line of any junction, or $b_i < +\infty$ and I_i is not the incoming transmission line of any junction, then a boundary data $\psi_i : [0, +\infty[\; \to \mathbb{R}$ is given. We ask ρ_i to satisfy $\rho_i(t, a_i) = \psi_i(t)$ (or $\rho_i(t, b_i) = \psi_i(t)$) in the sense of [5]. The treatment of boundary data in the sense of [5] can be done as in [1, 2], thus only the case without boundary data is considered. All the stated results hold also for the case with boundary data with obvious modifications.

Our aim is to solve the Cauchy problem on $[0, +\infty[$ for a given initial and boundary data as in next definition.

DEFINITION 3.5. *Given $\bar{\rho}_i : I_i \to [0, 1], i = 1, ..., N$, measurable functions, a collection of functions $\rho = (\rho_1, ..., \rho_N)$, with $\rho_i : [0, +\infty[\times I_i \to [0, 1]$ continuous as functions from $[0, +\infty[$ into L^1_{loc}, is an admissible solution to the Cauchy problem on the network if ρ_i is a weak entropic solution to (1.1) on $I_i, \rho_i(0, x) = \bar{\rho}_i \,(x)$ a.e., at each junction ρ is a weak solution and is an admissible weak solution in case of bounded variation.*

REMARK 3.6. *It is possible to generalize all definitions and results of next sections to the case of different fluxes f_i for each line I_i. In fact, all statements are in terms of values of fluxes at junctions, thus it is sufficient that the ranges of fluxes intersect.*

4. Riemann solvers at junctions. In this section we describe two different Riemann solvers at a junction that represent two different routing algorithms:

(RA1) We assume that

- (A) the traffic from incoming transmission lines is distributed on outgoing transmission lines according to fixed coefficients;
- (B) respecting (A) the router chooses to send packets in order to maximize fluxes (i.e., the number of packets which are processed).

(RA2) We assume that the number of packets through the junction is maximized both over incoming and outgoing lines.

Once solutions to Riemann problems are given, one can use a wave-front tracking algorithm to construct a sequence of approximate solutions. To pass to the limit one has to bound the number of waves and the BV norm of approximate solutions, see [7, 11]. In the next section we prove a BV bound on the density for the case of junctions with two incoming and two outgoing transmission lines, for both the routing algorithms.

4.1. RS for the algorithm (RA1).. The Riemann solver for the algorithm (RA1) has been already described in [10], [11] where traffic problems for road networks have been analyzed, using different assumptions on the flux function.

Consider a junction J in which there are n transmission lines with incoming traffic and m transmission lines with outgoing traffic. To deal with (A) we fix a traffic distribution matrix

$A \doteq \{\alpha_{ji}\}_{j=n+1,...,n+m, i=1,...,n} \in \mathbb{R}^{m\times n}$ such that

$$0 < \alpha_{ji} < 1, \quad \sum_{j=n+1}^{n+m} \alpha_{ji} = 1,$$

for each $i = 1, ..., n$ and $j = n+1, ..., n+m$, where α_{ji} is the percentage of packets arriving from the i-th incoming transmission line that take the j-th outgoing transmission line.

For simplicity we indicate by

$$(t, x) \in \mathbb{R}_+ \times I_i \ \rightarrow \rho_i(t, x) \in [0, 1], \quad i = 1, ..., n,$$

the densities of the packets on the transmission lines with incoming traffic and

$$(t, x) \in \mathbb{R}_+ \times I_j \ \rightarrow \rho_j(t, x) \in [0, 1], \quad j = n+1, ..., n+m,$$

those on transmission lines with outgoing traffic, see Figure 3.

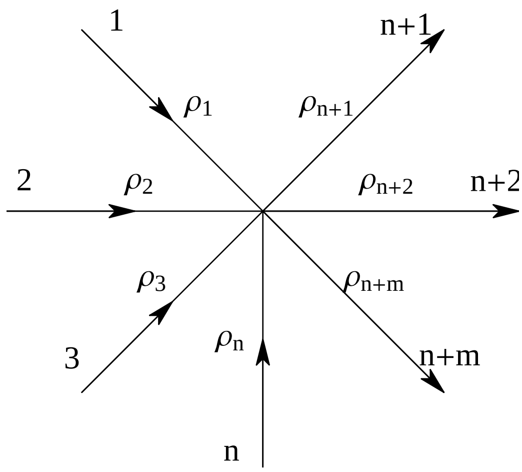

Fig. 4.1. *A junction.*

We need some more notations.

Definition 4.1. *Let $\tau : [0, 1] \ \rightarrow [0, 1]$ be the map such that:*

1. *$f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0, 1]$;*
2. *$\tau(\rho) \neq \rho$ for every $\rho \in [0, 1] \backslash \{\sigma\}$.*

Clearly, τ is well defined and satisfies

$$0 \le \rho \le \sigma \Leftrightarrow \sigma \le \tau(\rho) \le 1,$$
$$\sigma \le \rho \le 1 \Leftrightarrow 0 \le \tau(\rho) \le \sigma.$$

To state the main result of this section we need some assumption on the matrix A (satisfied under generic conditions for $m = n$). Let $\{e_1, ..., e_n\}$ be the canonical basis of $\mathbb{R}^n$ and for every subset $V \subset \mathbb{R}^n$ indicate by $V^\perp$ its orthogonal. Define for every $i = 1, ..., n$, $H_i = \{e_i\}^\perp$, i.e. the coordinate hyperplane orthogonal to e_i and for every $j = n+1, ..., n+m$ let $\alpha_j = \{\alpha_{j1}, ..., \alpha_{jn}\} \in \mathbb{R}^n$ and define $H_j = \{\alpha_j\}^\perp$. Let $\mathcal{K}$ be the set of indices $k = (k_1, ..., k_l), 1 \le l \le n-1$, such that $0 \le k_1 < k_2 < ... < k_l \le n+m$ and for every $k \in \mathcal{K}$ set $H_k = \bigcap_{h=1}^{l} H_h$. Letting $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^n$, we assume

(C) for every $k \in \mathcal{K}$, $\mathbf{1} \notin H_k^{\perp}$.

THEOREM 4.2. *(Theorem 3.1 of [11]) Consider a junction J, assume that the flux $f : [0,1] \to \mathbb{R}$ satisfies (F) and the matrix A satisfies condition (C). For every $\rho_{1,0}, ..., \rho_{n+m,0} \in [0,1]$, there exists a unique admissible centered weak solution, in the sense of Definition 3.4, $\rho = (\rho_1, ..., \rho_{n+m})$ of (1.1) at the junction J such that*

$$\rho_1(0,\cdot) \equiv \rho_{1,0}, ..., \rho_{n+m}(0,\cdot) \equiv \rho_{n+m,0}.$$

Moreover, there exists a unique $(n+m)$-tuple $(\hat{\rho}_1, ..., \hat{\rho}_{n+m}) \in [0,1]^{n+m}$ such that

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup \left]\tau(\rho_{i,0}), 1\right], & \text{if } 0 \leq \rho_{i,0} \leq \sigma, \\ [\sigma, 1], & \text{if } \sigma \leq \rho_{i,0} \leq 1, \end{cases} \quad i = 1, ..., n \tag{4.1}$$

and

$$\hat{\rho}_j \in \begin{cases} [0, \sigma], & \text{if } 0 \leq \rho_{j,0} \leq \sigma, \\ \{\rho_{j,0}\} \cup \left[0, \tau(\rho_{j,0})\right[, & \text{if } \sigma \leq \rho_{j,0} \leq 1, \end{cases} \quad j = n+1, ..., n+m, \tag{4.2}$$

and on each incoming line I_i, $i = 1, ..., n$, the solution consists of the single wave $(\rho_{i,0}, \hat{\rho}_i)$, while on each outgoing line I_j, $j = n+1, ..., n+m$, the solution consists of the single wave $(\hat{\rho}_j, \rho_{j,0})$.

Condition (C) on A can not hold for crossings with two incoming and one outgoing transmission lines. Following [10], it is possible to introduce a further parameter whose meaning is the following. When the number of packets is too big to let all of them go through crossing, there is a priority rule that describes the percentage of packets, going through the crossings, that comes from the first line. Since the construction happens to be a special case of that in next section, we omit details and refer the reader to [10] or to next section.

4.2. RS for the algorithm (RA2).. To solve Riemann problems according to (RA2) we need some additional parameters called priority and traffic distribution parameters. For simplicity of exposition, consider, first a junction J in which there are two transmission lines with incoming traffic and two transmission lines with outgoing traffic. In this case we have only one priority parameter $q \in]0,1[$ and one traffic distribution parameter $\alpha \in]0,1[$. We denote with $\rho_i(t,x), i = 1,2$ and $\rho_j(t,x), j = 3,4$ the traffic densities, respectively, on the incoming transmission lines and on the outgoing ones and by $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ the initial datum.

Define $\gamma_i^{\max}$ and $\gamma_j^{\max}$ as follows:

$$\gamma_i^{\max} = \begin{cases} f(\rho_{i,0}), & \text{if } \rho_{i,0} \in [0,\sigma], \\ f(\sigma), & \text{if } \rho_{i,0} \in \left]\sigma, 1\right], \end{cases} \quad i = 1,2, \tag{4.3}$$

and

$$\gamma_j^{\max} = \begin{cases} f(\sigma), & \text{if } \rho_{j,0} \in [0,\sigma], \\ f(\rho_{j,0}), & \text{if } \rho_{j,0} \in \left]\sigma, 1\right], \end{cases} \quad j = 3,4. \tag{4.4}$$

The quantities $\gamma_i^{\max}$ and $\gamma_j^{\max}$ represent the maximum flux that can be obtained by a single wave solution on each transmission line. In order to maximize the number of packets through the junction over incoming and outgoing lines we define

$$\Gamma = \min\left\{\Gamma_{in}^{\max}, \Gamma_{out}^{\max}\right\},$$

where $\Gamma_{in}^{\max} = \gamma_1^{\max} + \gamma_2^{\max}$ and $\Gamma_{out}^{\max} = \gamma_3^{\max} + \gamma_4^{\max}$. Thus we want to have Γ as flux through the junction.

Reasoning as in Theorem 4.2, one easily see that to solve the Riemann problem, it is enough to determine the fluxes $\hat{\gamma}_i = f(\hat{\rho}_i), i = 1,2$. In fact, to have simple waves with the appropriate velocities, i.e. negative on incoming lines and positive on outgoing ones, we get the constraints (4.1),(4.2). We have to distinguish two cases:

I $\Gamma_{in}^{\max} = \Gamma$,
II $\Gamma_{in}^{\max} > \Gamma$.

In the first case we set $\hat{\gamma}_i = \gamma_i^{\max}, i = 1, 2$.

Let us analyze the second case in which we use the priority parameter q. Not all packets can enter

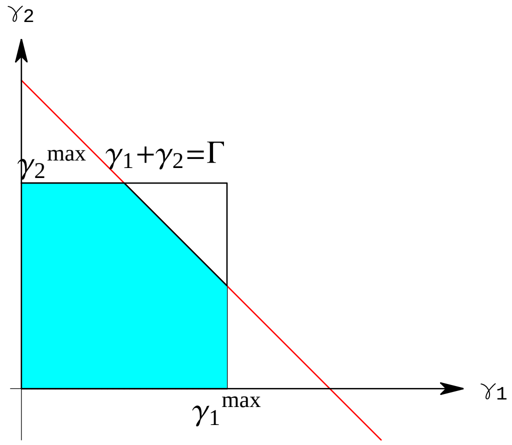

FIG. 4.2. *Case* $\Gamma_{in}^{\max} > \Gamma$.

the junction, so let C be the amount of packets that can go through. Then qC packets come from first incoming line and $(1-q)C$ packets from the second. Consider the space (γ_1, γ_2) and define the following lines:

$$r_q : \gamma_2 = \frac{1-q}{q}\gamma_1,$$

$$r_\Gamma : \gamma_1 + \gamma_2 = \Gamma.$$

Define P to be the point of intersection of the lines r_q and r_Γ. Recall that the final fluxes should belong to the region (see Figure 4.2):

$$\Omega = \{(\gamma_1, \gamma_2) : 0 \le \gamma_i \le \gamma_i^{\max}, i = 1, 2\}.$$

We distinguish two cases:

a) P belongs to Ω,

b) P is outside Ω.

In the first case we set $(\hat{\gamma}_1, \hat{\gamma}_2) = P$, while in the second case we set $(\hat{\gamma}_1, \hat{\gamma}_2) = Q$, with $Q = proj_{\Omega \cap r_\Gamma}(P)$ where *proj* is the usual projection on a convex set, see Figure 4.3.

The reasoning can be repeated also in the case of n incoming lines. In $\mathbb{R}^n$ the line r_q is given by $r_q = tv_q, t \in \mathbb{R}$, with $v_q \in \Delta_{n-1}$ where

$$\Delta_{n-1} = \left\{(\gamma_1, ..., \gamma_n) : \gamma_i \ge 0, i = 1, ..., n, \sum_{i=1}^{n} \gamma_i = 1\right\}$$

is the $(n-1)$ dimensional simplex and

$$H_\Gamma = \left\{(\gamma_1, ..., \gamma_n) : \sum_{i=1}^{n} \gamma_i = \Gamma\right\}$$

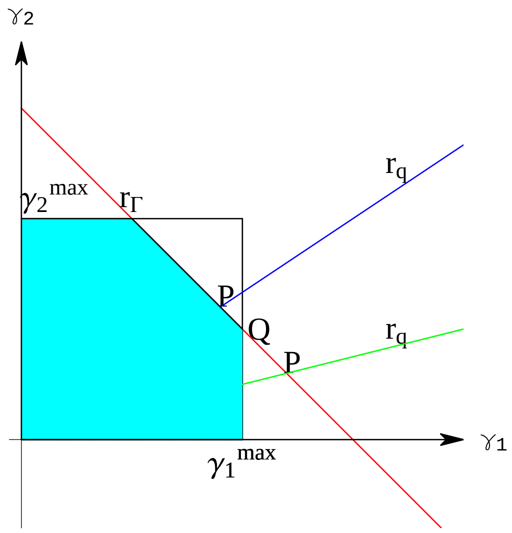

FIG. 4.3. *P belongs to Ω and P is outside Ω.*

is a hyperplane where $\Gamma = \min\{\sum_{in} \gamma_i^{\max}, \sum_{out} \gamma_j^{\max}\}$. Since $v_q \in \Delta_{n-1}$, there exists a unique point $P = r_q \cap H_\Gamma$. If $P \in \Omega$, then we set $(\hat{\gamma}_1, ..., \hat{\gamma}_n) = P$. If $P \notin \Omega$, then we set $(\hat{\gamma}_1, ..., \hat{\gamma}_n) = Q = proj_{\Omega \cap H_\Gamma}(P)$, the projection over the subset $\Omega \cap H_\Gamma$. Observe that the projection is unique since $\Omega \cap H_\Gamma$ is a closed convex subset of H_Γ.

REMARK 4.3. *A possible alternative definition in the case $P \notin \Omega$ is to set $(\hat{\gamma}_1, ..., \hat{\gamma}_n)$ as one of the vertices of $\Omega \cap H_\Gamma$.*

Let us now determine $\hat{\gamma}_j, j = 3, 4$. As for the incoming transmission lines we have to distinguish two cases :

I $\Gamma_{out}^{\max} = \Gamma$,
II $\Gamma_{out}^{\max} > \Gamma$.

In the first case $\hat{\gamma}_j = \gamma_j^{\max}, j = 3, 4$. Let us determine $\hat{\gamma}_j$ in the second case. Recall α the traffic distribution parameter. Since not all packets can go on the outgoing transmission lines, we let C be the amount that goes through. Then αC packets go on the outgoing line I_3 and $(1-\alpha)C$ on the outgoing line I_4.
Now we can proceed exactly as in the previous case with q replaced by α. More precisely, we define r_α by the equation $\gamma_4 = \frac{1-\alpha}{\alpha}\gamma_3$, r_Γ by $\gamma_3 + \gamma_4 = \Gamma$ and P to be the point of intersection of the lines r_α and r_Γ. Setting: $\Omega = \{(\gamma_3, \gamma_4) : 0 \le \gamma_j \le \gamma_j^{\max}, j = 3, 4\}$, we distinguish two cases:

a) P belongs to Ω
b) P is outside Ω.

In the first case we set $(\hat{\gamma}_3, \hat{\gamma}_4) = P$, while in the second case we set $(\hat{\gamma}_3, \hat{\gamma}_4) = Q$, where $Q = proj_{\Omega \cap r_\Gamma}(P)$. Again, we can extend to the case of m outgoing lines as for the incoming lines defining the hyperplane $H_\Gamma = \{(\gamma_{n+1}, \ldots, \gamma_{n+m}) : \sum_{j=n+1}^{n+m} \gamma_j = \Gamma\}$ and choosing a vector $v_\alpha \in \Delta_{m-1}$.

REMARK 4.4. *An alternative way of choosing the vector v_α is the following. We assume that a traffic distribution matrix A is assigned, then we compute $\hat{\gamma}_1, ..., \hat{\gamma}_n$ as before and choose $v_\alpha \in \Delta_{m-1}$ by*

$$v_\alpha = \Delta_{m-1} \cap \{tA(\hat{\gamma}_1, ..., \hat{\gamma}_n) : t \in \mathbb{R}\} .$$

The solution to Riemann problems in this section is consistent as shown by next Lemma.

Lemma 4.5. *(CC) holds for the Riemann Solver for (RA2) defined in this section.*

Proof. Let $\rho_0 = (\rho_{1,0}, \dots, \rho_{4,0})$ be the initial datum and $\hat{\rho} = RS(\rho_0)$. Assume, first, that $\Gamma < \Gamma_{in}^{\max}$. Define $\hat{\gamma}_i^{\max}$ to be the maximum flux on I_i given by a wave with left datum $\hat{\rho}_i$ and set then $\hat{\Gamma}_{in}^{\max} = \hat{\gamma}_1^{\max} + \hat{\gamma}_2^{\max}$. Then $\hat{\Gamma}_{in}^{\max} \geq \Gamma_{in}^{\max}$. Indeed if $\rho_{i,0} \in [0, \sigma[$ then $\hat{\rho}_i \in \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), \rho_{\max}]$ and $\hat{\gamma}_i^{\max} \geq \gamma_i^{\max} = f(\rho_{i,0})$. While if $\rho_{i,0} \in [\sigma, \rho_{\max}]$ then $\hat{\rho}_i \in [\sigma, \rho_{\max}]$ and so $\hat{\gamma}_i^{\max} = f(\sigma) = \gamma_i^{\max}$. The case $\Gamma < \Gamma_{out}^{\max}$ is treated similarly. □

5. Estimates on Density Variation. In this section we derive estimates on the total variation of the densities along a wave-front tracking approximate solution (constructed as in [11]) for both routing algorithms. This allows to construct the solutions to the Cauchy problem in standard way, see [7]. From now on, we assume that every junction has exactly two incoming transmission lines and two outgoing ones. This hypothesis is crucial, because the presence of more complicate junctions may provoke additional increases of the total variation of the flux and so of the density. The case where junctions have at most two incoming transmission lines and at most two outgoing ones can be treated in the same way.

From now on we fix a telecommunication network $(\mathcal{I}, \mathcal{J})$, with each node having at most two incoming and at most two outgoing lines, and a wave-front tracking approximate solution ρ, defined on the telecommunication network.

5.1. Algorithm (RA1). We first introduce the following:

Definition 5.1. *For every transmission line $I_i, i = 1, ..., N$, we indicate by*

$$\left(\rho_-^\beta, \rho_+^\beta\right), \quad \beta \in A = A(\rho, t, i), \quad A \text{ finite set},$$

the discontinuities on line I_i at time t, and by $x^\beta(t), \lambda^\beta(t), \beta \in A$, respectively their positions and velocities at time t. We also refer to the wave β to indicate the discontinuity $\left(\rho_-^\beta, \rho_+^\beta\right)$.

We have the following:

Lemma 5.2. *For some $K > 0$, we have*

$$\begin{aligned} TV(f(\rho(t, \cdot))) &\leq e^{Kt} TV(f(\rho(0+, \cdot))) \\ &\leq e^{Kt}(TV(f(\rho(0, \cdot))) + 2N f(\sigma)), \end{aligned}$$

for each $t \geq 0$, where N is the total number of transmission lines of the network. For the proof see Lemma 5.2 in [11]. To estimate the total variation of densities and to pass to the limit we need some additional notation.

Definition 5.3. *For every line I_i, we define two curves $Y_-^{i,\rho}(t), Y_+^{i,\rho}(t)$, called Boundary of External Flux, briefly BEF, in the following way. We set the initial condition $Y_-^{i,\rho}(0) = a_i, Y_+^{i,\rho}(0) = b_i$ (if $a_i = -\infty$, then $Y_-^{i,\rho} \equiv -\infty$ and if $b_i = +\infty$, then $Y_+^{i,\rho} \equiv +\infty$). We let $Y_\pm^{i,\rho}(t)$ follow the generalized characteristic as defined in [12], letting $Y_-^{i,\rho}(t) = a_i$ (resp. $Y_+^{i,\rho}(t) = b_i$) if the generalized characteristic reaches the boundary and $f'(\rho(t, a_i)) < 0$ (resp. $f'(\rho(t, b_i)) > 0$). (In this way $Y_\pm^{i,\rho}(t)$ may coincide with a_i or b_i for some time intervals). Let $\bar{t}$ be the first time such that $Y_-^{i,\rho}(\bar{t}) = Y_+^{i,\rho}(\bar{t})$ (possibly $\bar{t} = +\infty$), then we let $Y_\pm^{i,\rho}$ be defined on $[0, \bar{t}]$. Finally, we define the sets*

$$D_1^i(\rho) = \left\{(t, x) : t \in [0, \bar{t}[: Y_-^{i,\rho}(t) < x < Y_+^{i,\rho}(t)\right\},$$

and

$$D_2^i(\rho) = [0, +\infty[\times [a_i, b_i] \setminus D_1^i(\rho).$$

Clearly $Y_{\pm}^{i,\rho}(t)$ bound the set on which the datum is not influenced by other transmission lines through the junctions.

DEFINITION 5.4. *Fix a transmission line $I_i, i = 1, ..., N$ and a junction J. A wave β in I_i is said a big wave if*

$$sgn(\rho_-^{\beta} - \sigma) \cdot sgn(\rho_+^{\beta} - \sigma) \leq 0,$$

where $sgn(0) = 0$. We say that an incoming transmission line I_i has a bad datum at J at time $t > 0$ if

$$\rho_i(t, b_i-) \in [0, \sigma[\,,$$

while we say that an outgoing transmission line I_j has a bad datum at J at time $t > 0$ if

$$\rho_j(t, a_j+) \in \,]\sigma, 1]\,.$$

Our aim is now to bound, for each line I_i, the number of big waves inside the region $D_2^i(\rho)$, i.e. those generated by the influence of external lines.

LEMMA 5.5. *Let $\bar{t}$ be the time at which the two BEFs $Y_{\pm}^{i,\rho}$ interact. Assume $\bar{t} < +\infty$, $Y_{\pm}^{i,\rho}(\bar{t}) \in]a_i, b_i[$ and define*

$$\hat{\rho}_{out} = \rho\left(Y_{\pm}^{i,\rho}(\bar{t})-\right), \quad \hat{\rho}_{in} = \rho\left(Y_{\pm}^{i,\rho}(\bar{t})+\right), \quad \rho^* = \lim_{t \uparrow \bar{t}} \rho\left(Y_{-}^{i,\rho}(t)+\right) = \lim_{t \uparrow \bar{t}} \rho\left(Y_{+}^{i,\rho}(t)-\right).$$

If $\hat{\rho}_{in}$, respectively $\hat{\rho}_{out}$, is a bad datum for I_i as incoming line, respectively for I_i as outgoing line, then there exists no value ρ^* of the density such that

$$\lambda(\hat{\rho}_{out}, \rho^*) > \lambda(\rho^*, \hat{\rho}_{in}).$$

Proof. Since $\hat{\rho}_{out}$ and $\hat{\rho}_{in}$ are bad data for, respectively, an outgoing transmission line and an incoming transmission line, it follows that

$$\hat{\rho}_{out} \in \,]\sigma, 1]\,, \quad \hat{\rho}_{in} \in [0, \sigma[\,.$$

Observe that $\hat{\rho}_{out}$ and ρ^* must be connected by a single wave, thus $\rho^* \geq \sigma$, otherwise the wave would be split in a fan of rarefaction shocks.

Similarly, ρ^* and $\hat{\rho}_{in}$ must be connected by a single wave, thus $\rho^* \leq \sigma$, otherwise the wave would be split in a fan of rarefaction shocks.

Finally, $\rho^* = \sigma$, but then

$$\lambda(\hat{\rho}_{out}, \rho^*) \leq 0 \leq \lambda(\rho^*, \hat{\rho}_{in})$$

and the conclusion holds. □

LEMMA 5.6. *For every $t \geq 0$, there are at most two big waves on*

$$\left\{x : (t, x) \in D_2^i(\rho)\right\} \subseteq [a_i, b_i]\,.$$

Proof. A big wave can originate at time t on transmission line I_i from J only if the line I_i has a bad datum at J at time t. If this happens, then, from Theorem 4.2, line I_i has not a bad datum at J up to the time in which a big wave is absorbed from I_i. This concludes the proof if $D_2^i(\rho)$ is formed by two connected components.

It remains to consider the time at which the two BEFs interact. By Lemma 5.5 we have that not both connected components can contain a big wave. Thus again there are at most two big waves. □

Up to now, we did not make use of assumption (F), which is necessary for next Lemma:

LEMMA 5.7. *Assume (F), then for some $K > 0$, we have*

$$TV(\rho(t,\cdot)) \le TV(\rho(0,\cdot)) + 2N\left(\frac{e^{Kt}f(\sigma)}{\bar{v}} + 1\right),$$

for each $t \ge 0$, where N is the total number of transmission lines of the network.

Proof. Let $TV(h;[a,b])$ denote the total variation of the function h over the interval $[a,b]$ and define

$$TV^j(\rho(t)) = \sum_i TV(\rho(t); D^i_j(\rho(t))), \qquad j = 1,2,$$

which are, respectively, the total variation of $\rho(t)$ due to the evolution only inside each line I_i and by interaction with junctions. Clearly:

$$TV(\rho(t)) = TV^1(\rho(t)) + TV^2(\rho(t)).$$

Since $D^i_1(\rho(t))$ is not influenced by external lines, we are in the situation of a conservation law on $\mathbb{R}$, hence

$$TV^1(\rho(t)) \le TV(\rho(0)).$$

Let $B(t)$ denote the number of big waves generated from junctions, i.e. the number of big waves in $\cup_i D^i_2(\rho(t))$. Then by chain rule for BV functions and Lemma 5.2:

$$TV^2(\rho(t)) \le \frac{1}{\bar{v}} TV^2(f(\rho(t)) + B(t) \le \frac{1}{\bar{v}} e^{Kt}(TV^2(f(\rho(0+))) + B(t). \tag{5.1}$$

Now $TV^2(\rho(0)) = 0$, thus, using again Lemma 5.2 and Lemma 5.6, the following relation holds:

$$TV^2(\rho(t)) \le \frac{1}{\bar{v}} e^{Kt} 2N f(\sigma) + 2N. \tag{5.2}$$

Finally we get

$$TV(\rho(t)) = TV^1(\rho(t)) + TV^2(\rho(t)) \le TV(\rho(0)) + 2N\left(\frac{e^{Kt}f(\sigma)}{\bar{v}} + 1\right).$$

□

Thanks to Lemma 5.7 and the Lipschitz continuous dependence in L^1_{loc} of wave-front tracking approximations, we can apply Helly Theorem, as in [7] to get existence of solutions:

THEOREM 5.8. *Fix a telecommunication network (I, J) and assume (F). Given $T > 0$, for every initial data there exists an admissible solution to the Cauchy problem on the network defined on $[0, T]$.*

Let us observe that there is no a Lipschitz continuous dependence by initial data with respect to the L^1 norm. In fact it is possible to choose two piecewise constant initial data, which are exactly the same except for a shift of a discontinuity, such that the L^1-distance of the two corresponding solutions increases by an arbitrary multiplicative factor (see [11]).

5.2. Algorithm (RA2). Let us now estimate the flux total variation and the density total variation for the routing algorithm (RA2). We can define BEFs, bad datum and big waves as in the previous section.

Fix a junction J with two incoming transmission lines I_1 and I_2 and two outgoing ones I_3 and I_4.

Suppose that at some time $\bar{t}$ a wave interacts with the junction J and let $(\rho_1^-, \rho_2^-, \rho_3^-, \rho_4^-)$ and $(\rho_1^+, \rho_2^+, \rho_3^+, \rho_4^+)$ indicate the equilibrium configurations at the junction J before and after the interaction respectively. Introduce the following notation

$$\gamma_i^\pm = f(\rho_i^\pm), \quad \Gamma_{in}^\pm = \gamma_{1,\max}^\pm + \gamma_{2,\max}^\pm, \quad \Gamma_{out}^\pm = \gamma_{3,\max}^\pm + \gamma_{4,\max}^\pm,$$

$$\Gamma^\pm = \min\{\Gamma_{in}^\pm, \Gamma_{out}^\pm\},$$

where $\gamma_{i,\max}^\pm, i = 1,2$ and $\gamma_{j,\max}^\pm, j = 3,4$ are defined as in (4.3) and (4.4). In general $-$ and $+$ denote the values before and after the interaction, while by Δ we indicate the variation, i.e. the value after the interaction minus the value before. For example $\Delta\Gamma = \Gamma^+ - \Gamma^-$. Let us denote by $TV(f)^\pm = TV(f(\rho(\bar{t}\pm, \cdot)))$ the flux variation of waves before and after the interaction, and

$$TV(f)_{in}^\pm = TV(f(\rho_1(\bar{t}\pm, \cdot))) + TV(f(\rho_2(\bar{t}\pm, \cdot))),$$

$$TV(f)_{out}^\pm = TV(f(\rho_3(\bar{t}\pm, \cdot))) + TV(f(\rho_4(\bar{t}\pm, \cdot))),$$

the flux variation of waves before and after the interaction, respectively, on incoming and outgoing lines.

Let us prove some estimates which are used later to control the total variation of the density function. For simplicity, from now on we assume that:

(A) the wave interacting at time $\bar{t}$ with J comes from line 1 and we let ρ_1 be the value on the left of the wave.

The case of a wave from an outgoing line can be treated similarly.

Lemma 5.9. *We have*

$$sgn\,(\Delta\gamma_3) \cdot sgn\,(\Delta\gamma_4) \geq 0.$$

Proof. To prove the lemma it is enough to observe that a variation of γ_3 is due to a movement along the line r_q or along $\gamma_3 = c_1$ or $\gamma_4 = c_2$ with c_1 and c_2 constant. In each case $\Delta\gamma_3$ and $\Delta\gamma_4$ have the same sign. □

In the same way we can prove the following Lemma:

Lemma 5.10. *We have*

$$sgn(\gamma_1^+ - \gamma_1) \cdot sgn(\Delta\gamma_2) \geq 0,$$

where $\gamma_1 = f(\rho_1)$.

Lemma 5.11. *It holds*

$$TV(f)_{out}^+ = |\Delta\Gamma|.$$

Proof. To prove the lemma it is enough to observe that

$$\Gamma^- = \gamma_3^- + \gamma_4^-, \quad \Gamma^+ = \gamma_3^+ + \gamma_4^+,$$

$$|\Delta\Gamma| = |\Gamma^+ - \Gamma^-| = |(\gamma_3^+ - \gamma_3^-) + (\gamma_4^+ - \gamma_4^-)|$$

from which, by Lemma 5.9, we have

$$|\Delta\Gamma| = |\Delta\gamma_3| + |\Delta\gamma_4| = TV(f)^+_{out}.$$

□

LEMMA 5.12. *We have*

$$TV(f)^-_{in} = TV(f)^+_{in} + |\Delta\Gamma|. \tag{5.3}$$

Proof. Clearly since the wave on the first line has positive velocity, we have $0 \le \rho_1 \le \sigma$. Since $\rho_1 \le \sigma$, observe that the maximum flux for ρ_1^+, which is the solution with initial data ρ_1, is given by $\gamma_{1,\max} = f(\rho_1)$. Also

$$TV(f)^- = TV(f)^-_{in} = |\gamma_1 - \gamma_1^-|.$$

We have two possibilities:
Case 1) $\rho_1^- \le \sigma$,
Case 2) $\rho_1^- > \sigma$.

Let us first analyze Case 1). Then we further split it into two subcases:
Case 1a) $\rho_1 < \rho_1^-$,
Case 1b) $\rho_1 > \rho_1^-$.

If 1a) holds true, since $\rho_1 < \rho_1^-$, we get $\gamma_{1,\max} = f(\rho_1) < f(\rho_1^-) = \gamma_{1,\max}^-$ and one of the following holds:
Case 1a.1) $\Gamma^- = \Gamma^-_{in}$,
Case 1a.2) $\Gamma^- = \Gamma^-_{out}$.

In Case 1a.1) from $\gamma_{1,\max} < \gamma_{1,\max}^-$ and $\Gamma^- = \Gamma^-_{in}$, it follows that $\Gamma^+ = \Gamma^+_{in}$, from which $\gamma_2^+ = \gamma_2^-$, $\gamma_1^+ = \gamma_1$ and then $TV(f)^+_{in} = 0$.

In the other Case 1a.2) we have $\gamma_{1,\max} < \gamma_{1,\max}^-$, hence $\Gamma^-_{in} \ge \Gamma^-$ and $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma^-_{in}$. The following distinction must be considered:
Case 1a.2.1) $\gamma_{1,\max} + \gamma_{2,\max}^- \ge \Gamma^-$,
Case 1a.2.2) $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma^-$.

If Case 1a.2.1) holds, from $\gamma_{1,\max} + \gamma_{2,\max}^- \ge \Gamma^-$, we have that $\Gamma^+ = \Gamma^-$, from which $|\Delta\Gamma| = 0$. By Lemma 5.10 the conclusion holds.

In the opposite Case 1a.2.2) from $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma^-$, one gets $\Gamma^+ = \gamma_{1,\max} + \gamma_{2,\max}^-$, from which it follows that $\mathrm{T}V(f)^+_{in} = 0$. Then $|\Delta\Gamma| = \left|\gamma_1^- - \gamma_1\right| = TV(f)^-_{in}$. Case 1a) is thus finished.

Let us now focus on Case 1b). We have to distinguish two possibilities:
Case 1b.1) $\Gamma^- = \Gamma^-_{out}$,
Case 1b.2) $\Gamma^- = \Gamma^-_{in}$.

If Case 1.b.1) holds, from $\Gamma^- = \Gamma^-_{out}$ it follows that $\gamma_{1,\max} + \gamma_{2,\max}^- > \Gamma^-_{in}$. Then $\Gamma^+ = \Gamma^-$, hence $|\Delta\Gamma| = 0$ and by Lemma 5.10 the conclusion holds.

In Case 1.b.2), we have $\gamma_{1,\max} + \gamma_{2,\max}^- > \Gamma^-_{in}$ and $\Gamma^-_{out} \ge \Gamma^-_{in}$ and following cases may happen:
Case 1b.2.1) $\gamma_{1,\max} + \gamma_{2,\max}^- \le \Gamma^-_{out}$,
Case 1b.2.2) $\gamma_{1,\max} + \gamma_{2,\max}^- > \Gamma^-_{out}$.

Consider Case 1b.2.1) first. From $\gamma_{1,\max} + \gamma_{2,\max}^- \le \Gamma^-_{out}$, one has $TV(f)^+_{in} = 0$, hence $|\Delta\Gamma| = \left|\gamma_1 - \gamma_1^-\right| = TV(f)^-_{in}$.

In Case 1b.2.2), from $\gamma_{1,\max} + \gamma_{2,\max}^- > \Gamma^-_{out}$ we obtain $\Gamma^+ = \Gamma^+_{out}$. By Lemma 5.9,

$$TV(f)^+_{in} = \gamma_{1,\max} + \gamma_{2,\max}^- - \Gamma^-_{out},$$

$$TV(f)^-_{in} = \gamma_{1,\max} - \gamma^-_{1,\max},$$

hence

$$TV(f)^-_{in} - TV(f)^+_{in} = -\gamma^-_{1,\max} - \gamma^-_{2,\max} + \Gamma^-_{out} =$$
$$= \Gamma^+ - \Gamma^-_{in} = \Gamma^+ - \Gamma^- = |\Delta\Gamma|\,.$$

Let us analyze Case 2). Since $\rho_1^- > \sigma$ it follows that $\rho_1 < \tau(\rho_1^-) < \sigma$. Observe that $\gamma_1 = f(\rho_1) < f(\rho_1^-) = \gamma_1^-$ and $\gamma^-_{1,\max} = f(\sigma), \gamma_{1,\max} = f(\rho_1)$.

We have to distinguish two cases:

Case 2a) $\Gamma^- = \Gamma^-_{in}$,

Case 2b) $\Gamma^- = \Gamma^-_{out}$.

If Case 2a) holds, then one gets $\gamma_{1,\max} + \gamma^-_{2,\max} < \Gamma^-$, from which it follows that $\Gamma^+ = \gamma_{1,\max} + \gamma^-_{2,\max}$. Hence $TV(f)^+_{in} = 0$ and the conclusion holds.

For the opposite Case 2b), we have $\gamma_{1,\max} + \gamma^-_{2,\max} < \Gamma^-_{in}$ and $\Gamma^-_{in} \geq \Gamma^-_{out}$. Hence the following two cases are possible:

Case 2b.1) $\gamma_{1,\max} + \gamma^-_{2,\max} \geq \Gamma^-_{out}$,

Case 2b.2) $\gamma_{1,\max} + \gamma^-_{2,\max} < \Gamma^-_{out}$.

In Case 2.b.1), from $\gamma_{1,\max} + \gamma^-_{2,\max} \geq \Gamma^-_{out}$, it follows that $\Gamma^+ = \Gamma^-$. The latter implies $|\Delta\Gamma| = 0$ and the conclusion follows from Lemma 5.10.

In Case 2.b.2) from $\gamma_{1,\max} + \gamma^-_{2,\max} < \Gamma^-_{out}$, we obtain $\Gamma^+ = \gamma_{1,\max} + \gamma^-_{2,\max}$. Thus, by Lemma 5.10, we get:

$$TV(f)^+_{out} = \Gamma^+ - (\gamma_1 + \gamma_2^-) = (\gamma_{1,\max} + \gamma^-_{2,\max}) - (\gamma_1 + \gamma_2^-) =$$
$$= \gamma^-_{2,\max} - \gamma_2^-\,.$$

It follows that

$$|\Delta\Gamma| = \Gamma^- - \Gamma^+ = \gamma_1^- + \gamma_2^- - (\gamma_{1,\max} + \gamma^-_{2,\max})$$
$$= (\gamma_1^- - \gamma_{1,\max}) + (\gamma_2^- - \gamma^-_{2,\max}) = TV(f)^-_{in} - TV(f)^+_{out},$$

and the conclusion holds. The proof is thus finished. □

From the above results, we are ready to state the following:

LEMMA 5.13. *The flux variation $TV(f)$ is conserved along wave-front tracking approximations.* Notice that this result is much stronger than that obtained for routing algorithm (RA1), for which only an exponential in time bound for the flux variation is achieved.

Proof. From Lemma 5.11 and Lemma 5.12 we get

$$TV(f)^- = TV(f)^-_{in} = TV(f)^+_{in} + |\Delta\Gamma| = TV(f)^+.$$

□

The estimate on the number of big waves is valid also for the algorithm (RA2), thus we bound the total variation of the densities as follows.

THEOREM 5.14. *Consider a telecommunication network $(\mathcal{I}, \mathcal{J})$ and assume (F). Let ρ be a wave-front tracking approximate solution, then*

$$TV(\rho(t,\cdot)) \leq TV(\rho(0,\cdot)) + 2N\left(\frac{f(\sigma)}{\bar{v}} + 1\right),$$

for each $t \geq 0$, where N is the total number of transmission lines of the network. Moreover given $T > 0$, there exists an admissible solution to the Cauchy problem on the network defined on $[0,T]$ for every initial data.

5.2.1. Uniqueness and Lipschitz continuous dependence. The aim of this section is to prove Lipschitz continuous dependence by initial data for solutions to the Cauchy problem on the network, controlling for any two approximate solutions ρ, ρ' how their distance varies in time. We use the method introduced in [8], which is based on a Riemannian type distance on L^1. There are various alternative methods to treat uniqueness and continuous dependence for the case of scalar conservation laws on the real line, among which: Kruzkov entropies (cfr. [7]), viscous approximations (cfr. [22]) and Bressan-Liu-Yang functionals (see [9]). No one of these methods seems to work for the network case. In fact, Kruzkov method requires to estimate integrals on a region in $\mathbb{R}^2$, which now is replaced by an integral on the topological space obtained by the product of the network and $\mathbb{R}$. On the other side, it is not clear how to define a viscous solutions on the network, in particular how to treat boundary data at nodes, and how to pass to the limit. Finally, a Bressan-Liu-Yang type functional requires to introduce a definition of approaching waves, but, on a general network, with complicate topology, every wave is potentially approaching each other.

The basic idea is to estimate the L^1-distance viewing L^1 as a Riemannian manifold. We consider the subspace of piecewise constant functions and "generalized tangent vectors" consisting of two components (v, ξ), where $v \in L^1$ describes the L^1 infinitesimal displacement, while $\xi \in \mathbb{R}^n$ describes the infinitesimal displacement of discontinuities. For example, take a family of piecewise constant functions $\theta \to \rho^\theta$, $\theta \in [0,1]$, each of which has the same number of jumps, say at the points $x_1^\theta < ... < x_N^\theta$. Assume that the following functions are well defined (Fig. 6)

$$L^1 \ni v^\theta(x) \doteq \lim_{h\to 0} \frac{\rho^{\theta+h}(x) - \rho^\theta(x)}{h},$$

and also the numbers

$$\xi_\beta^\theta \doteq \lim_{h\to 0} \frac{x_\beta^{\theta+h} - x_\beta^\theta}{h}, \qquad \beta = 1, ..., N.$$

Then we say that γ admits tangent vectors $(v^\theta, \xi^\theta) \in T_{\rho^\theta} \doteq L^1(\mathbb{R};\mathbb{R}^n) \times \mathbb{R}^n$. In general such path

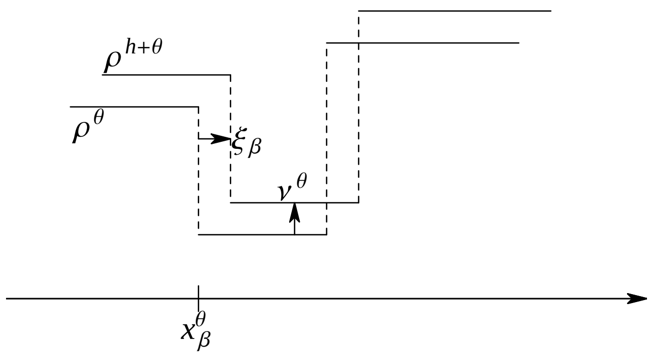

FIG. 5.1. *Construction of "generalized tangent vectors".*

$\theta \to \rho^\theta$ is not differentiable w.r.t. the usual differential structure of L^1, in fact if $\xi_\beta^\theta \neq 0$, as $h \to 0$ the ratio $\left[\rho^{\theta+h}(x) - \rho^\theta\right]/h$ does not converge to any limit in L^1.
Moreover, we can compute the L^1-length of the path $\gamma : \theta \to \rho^\theta$ in the following way:

$$\|\gamma\|_{L^1} = \int_0^1 \left\|v^\theta\right\|_{L^1} d\theta + \sum_{\beta=1}^N \int_0^1 \left|\rho^\theta(x_\beta+) - \rho^\theta(x_\beta-)\right| \left|\xi_\beta^\theta\right| d\theta. \tag{5.4}$$

According to (5.4), in order to compute the L^1-length of a path γ, we integrate the norm of its tangent vector which is defined as follows:

$$\|(v,\xi)\| \doteq \|v\|_{L^1} + \sum_{\beta=1}^{N} |\Delta\rho_\beta|\,|\xi_\beta|\,,$$

where $\Delta\rho_\beta = \rho(x_\beta+) - \rho(x_\beta-)$ is the jump across the discontinuity x_β.

Let us introduce the following definition.

Definition 5.15. *We say that a continuous map $\gamma : \theta \to \rho^\theta \doteq \gamma(\theta)$ from $[0,1]$ into L^1_{loc} is a regular path if the following holds. All functions ρ^θ are piecewise constant, with the same number of jumps, say at $x_1^\theta < ... < x_N^\theta$ and coincide outside some fixed interval $]-M,M[$. Moreover, the function $\theta \to \rho_x^\theta$ is continuous from $[0,1]$ into L^1, and the map $\theta \to \rho^\theta$ admits a generalized tangent vector $D\gamma(\theta) = (v^\theta, \xi^\theta) \in T_{\gamma(\theta)} = L^1(\mathbb{R};\mathbb{R}^n) \times \mathbb{R}^N$, continuously depending on θ .* Given two piecewise constant functions ρ and ρ', call $\Omega(\rho,\rho')$ the family of all regular paths $\gamma : [0,1] \to \gamma(t)$ with $\gamma(0) = \rho$, $\gamma(1) = \rho'$. The Riemannian distance between ρ and ρ' is given by

$$d(\rho,\rho') \doteq \inf\left\{\|\gamma\|_{L^1}, \gamma \in \Omega(\rho,\rho')\right\}.$$

To define d on all L^1, for given $\rho, \rho' \in L^1$ we set

$$\begin{aligned} d(\rho,\rho') \doteq \inf\{&\|\gamma\|_{L^1} + \|\rho - \tilde\rho\|_{L^1} + \|\rho' - \tilde\rho'\|_{L^1} : \\ &\tilde\rho, \tilde\rho' \text{ piecewise constant functions}, \gamma \in \Omega(\tilde\rho, \tilde\rho')\}. \end{aligned}$$

It is easy to check that this distance coincides with the distance of L^1. For the systems case, one has to introduce weights, see [8].

Now we are ready to estimate the L^1 distance among solutions, studying the evolution of norms of tangent vectors along wave-front tracking approximations. Take ρ, ρ' piecewise constant functions and let $\gamma_0(\vartheta) = \rho^\vartheta$ be a regular path joining $\rho = \rho^0$ with $\rho' = \rho^1$. Define $\rho^\vartheta(t,x)$ to be a wave-front tracking approximate solution with initial data ρ^ϑ and let $\gamma_t(\vartheta) = \rho^\vartheta(t,\cdot)$.

If we can prove that, for every γ_0 (regular path) and every $t \geq 0$, γ_t is a regular path and

$$\|\gamma_t\|_{L^1} \leq \|\gamma_0\|_{L^1}\,, \tag{5.5}$$

then for every $t \geq 0$

$$\|\rho(t,\cdot) - \rho'(t,\cdot)\|_{L^1} \leq \inf_{\gamma_t} \|\gamma_t\|_{L^1} \leq \inf_{\gamma_0} \|\gamma_0\|_{L^1} = \|\rho(0,\cdot) - \rho'(0,\cdot)\|_{L^1}\,. \tag{5.6}$$

To obtain (5.5), hence (5.6), it is enough to prove that, for every tangent vector $(v,\xi)(t)$ to any regular path γ_t, one has:

$$\|(v,\xi)(t)\| \leq \|(v,\xi)(0)\|\,, \tag{5.7}$$

i.e the norm of a tangent vector does not increase in time. Moreover, if (5.6) is established, then uniqueness and Lipschitz continuous dependence of solutions to Cauchy problems is straightforwardly achieved passing to the limit on the wave-front tracking approximate solutions.

The same reasoning can be used on the network. If $\rho = (\rho_1, ..., \rho_N)$ is a solution on the network then we set

$$\|\rho\|_{L^1} = \sum_i \|\rho_i\|_{L^1(I_i)}.$$

To estimate the distance among wave-front tracking solutions it is thus enough to prove (5.7). We prove the latter estimating the evolution of the tangent vector norm at each time. For this, we fix a time $\bar t \geq 0$ and, without loss of generality, treat the following cases:

a) no interaction of waves takes place in any transmission line at $\bar{t}$ and no wave interacts with a junction;
b) two waves interact at $\bar{t}$ on a transmission line and no other interaction takes place;
c) a wave interacts with a junction at $\bar{t}$ and no other interaction takes place.

In case a) we can prove

$$\left[\frac{d}{dt}\|(v,\xi)(t)\|\right]_{t=\bar{t}} \le 0,$$

while in cases b) and c), letting $(v,\xi)^{\pm}$ be the tangent vector before $(-)$ and after $(+)$ the interaction, we prove

$$\left\|(v,\xi)^{+}\right\| \le \left\|(v,\xi)^{-}\right\|.$$

Let us first analyze the case a). Denote by x_β, σ_β, and ξ_β, respectively, the positions, sizes and shifts of the discontinuities of the wave-front tracking approximate solution. Following [8] we get:

$$\frac{d}{dt}\left\{\int |v(t,x)|\,dx + \sum_{\beta=1}^{N} |\xi_\beta|\,|\sigma_\beta|\right\} =$$
$$-\left\{\sum_{\beta}\left(\lambda(\rho^-) - \dot{x}_\beta\ \right)\left|v^-\right| + \sum_{\beta}\left(\dot{x}_\beta - \lambda(\rho^+)\right)\left|v^+\right|\right\} +$$
$$+\sum_{\beta} D\lambda(\rho^-,\rho^+)\cdot(v^-,v^+)\,(\mathrm{sign}\xi_\beta)\,|\sigma_\beta|\,,$$

with $\sigma_\beta = \rho^+ - \rho^-$, $\rho^{\pm} \doteq \rho(x_\beta\pm)$ and similarly for $v^{\pm}$. If the waves respect the Rankine-Hugoniot conditions, then

$$D\lambda(\rho^-,\rho^+)(v^-,v^+) = \left(\lambda(\rho^-) - \dot{x}_\beta\right)\frac{v^-}{|\sigma_\beta|} + \left(\dot{x}_\beta - \lambda(\rho^+)\right)\frac{v^+}{|\sigma_\beta|}$$

and

$$\frac{d}{dt}\left\{\int |v(t,x)|\,dx + \sum_{\beta=1}^{N} |\xi_\beta|\,|\sigma_\beta|\right\} \le 0. \tag{5.8}$$

Remark 5.16. *To be precise, to obtain a control on $TV(f)$ the wave-front tracking is slightly modified in the following way, see [11]. For every initial data ρ a sequence of piecewise constant approximations ρ_ν are constructed, converging to ρ in L^1. Then one choose a sequence $\delta_\nu > 0$ converging to zero and construct wave-front tracking approximate solutions splitting rarefaction waves into a fan of rarefaction shocks, each of size at most δ_ν. If a rarefaction wave is originated at a junction with ρ^+ or ρ^- equal to σ, then we let $\dot{x}_\beta = 0$. However, since $\dot{x}_\beta = \frac{f(\sigma+\delta_\nu)-f(\sigma)}{\delta_\nu}$, $|\dot{x}_\beta - \bar{x}_\beta| = \delta_\nu$ we get*

$$\frac{d}{dt}\left\{\int |v(t,x)|\,dx + \sum_{\beta=1}^{N} |\xi_\beta|\,|\sigma_\beta|\right\} \le 2\delta_\nu N,$$

where N is the number of transmission lines. In fact, by Lemma 5.6, there are at most two such waves on each transmission line. Hence the estimate (5.7) is obtained in the limit as ν tends to $+\infty$.

In case (b), we use the following Lemma (see [11] for example):

LEMMA 5.17. *Let us consider in a transmission line two waves, with speeds λ_1 and λ_2 respectively, that interact producing a wave with speed λ_3. If the first wave is shifted by ξ_1 and the second wave by ξ_2, then the shift of the resulting wave is given by*

$$\xi_3 = \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_2}\xi_1 + \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}\xi_2.$$

Moreover we have that

$$\Delta\rho_3\xi_3 = \Delta\rho_1\xi_1 + \Delta\rho_2\xi_2, \tag{5.9}$$

where $\Delta\rho_i$ are the signed strengths of the corresponding waves. From (5.9) it follows

$$|\Delta\rho_3\xi_3| \le |\Delta\rho_1|\,|\xi_1| + |\Delta\rho_2|\,|\xi_2|\,,$$

from which

$$\left\|(v,\xi)^+\right\| \le \left\|(v,\xi)^-\right\|. \tag{5.10}$$

For case c) we report the lemma in [11]:

LEMMA 5.18. *Let us consider a junction J with incoming lines I_1 and I_2 and outgoing lines I_3 and I_4. If a wave on a transmission line I_i $(i \in \{1,...,4\})$ interacts with J and if ξ_i is the shift of the wave in I_i, then the shift ξ_j produced in a different line I_j $(j \in \{1,...,4\}\backslash\{i\})$ satisfies*

$$\xi_j(\rho_j^+ - \rho_j^-) = \frac{\Delta\gamma_j}{\Delta\gamma_i}\xi_i(\rho_i^+ - \rho_i^-),$$

where $\Delta\gamma_l(l \in \{i,j\})$ represents the variation of the flux in the line I_l and $\rho_l^-, \rho_l^+(l \in \{i,j\})$ are the states at J in the line I_l respectively before and after the interaction.

Define $TV(f)^{\pm}$ to be the total variation of the flux of the solution before $(-)$ and after $(+)$ the interaction, and $TV(f)_i^{\pm}$ the same quantity on line I_i. Without loss of generality, we can assume that a wave from an incoming transmission line $\bar{\imath}$ interacts with a junction J and no other wave is present. Then $TV(f)^- = TV(f)_{\bar{\imath}}^-$ and $TV(f)^+ = \sum_j TV(f)_j^+$ where $TV(f)_j^+$ measures just the wave produced by the interaction. From Lemma 5.18 we have

$$|\xi_j|\,|\Delta\rho_j| = \frac{TV(f)_j^-}{TV(f)^-}\,|\xi_i|\,|\Delta\rho_i|\,.$$

Using Lemma 5.13 we conclude

$$\begin{aligned} \|(v,\xi)^+\| &= \|v\|_{L^1} + \textstyle\sum_j |\xi_j|\,|\Delta\rho_j| = \|v\|_{L^1} + \sum_j \frac{TV(f)_j^-}{TV(f)_{\bar{\imath}}^-}\,|\xi_i|\,|\Delta\rho_i| \\ &= \|v\|_{L^1} + \tfrac{TV(f)^+}{TV(f)^-}\,|\xi_i|\,|\Delta\rho_i| = \|(v,\xi)^-\|\,. \end{aligned} \tag{5.11}$$

From (5.8),(5.10) and (5.11), we get the following:

THEOREM 5.19. *Consider a telecommunication network $(\mathcal{I},\mathcal{J})$ and assume (F). Then the solutions to Cauchy problems on the networks are unique and depend in a Lipschitz continuous way from initial data.*

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