Traffic Flow on a Road Network

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Abstract

This paper is concerned with a fluidodynamic model for traffic flow. More precisely, we consider a single conservation law, deduced from the conservation of the number of cars, defined on a road network that is a collection of roads with junctions. The evolution problem is underdetermined at junctions, hence we choose to have some fixed rules for the distribution of traffic plus an optimization criteria for the flux. We prove existence of solutions to the Cauchy problem and we show that the Lipschitz continuous dependence by initial data does not hold in general.

Our method is based on wave front tracking approach, see [6], and works also for boundary data and time dependent coefficients of traffic distribution at junctions, so including traffic lights.

Key Words: scalar conservation laws, traffic flow.
AMS Subject Classifications: 90B20, 35L65.

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1 Introduction.

This paper deals with a fluidodynamic model of heavy traffic on a road network. More precisely, we consider the conservation law formulation proposed by Lighthill and Whitham [14] and Richards [15]. This nonlinear framework is based simply on the conservation of cars and is described by the equation:

$$\rho_t + f(\rho)_x = 0, \tag{1.1}$$

where $\rho = \rho(t,x) \in [0, \rho_{max}]$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, is the *density* of cars, $v = v(t,x)$ is the *velocity* and $f(\rho) = v\,\rho$ is the *flux*. This model is appropriate to reveal shocks formation as it is natural for conservation laws, whose solutions may develop discontinuities in finite time even for smooth initial data (see [6]). In most cases one assumes that v is a function of ρ only and that the corresponding flux is a concave function. We make this assumption, moreover we let f have a unique maximum $\sigma \in]0, \rho_{max}[$ and for notational simplicity we assume $\rho_{max} = 1$.

Here we deal with a network of roads, as in [12]. This means that we have a finite number of roads modeled by intervals $[a_i, b_i]$ (with one of the two endpoints possibly infinite) that meet at some junctions. For endpoints that do not touch a junction (and are not infinite), we assume to have a given boundary data and solve the corresponding boundary problem, as in [1, 2, 3, 5]. The key role is played by junctions at which the system is underdetermined even after prescribing the conservation of cars, that can be written as the Rankine-Hugoniot relation:

$$\sum_{i=1}^{n} f(\rho_i(t,b_i)) = \sum_{j=n+1}^{n+m} f(\rho_j(t,a_j)), \tag{1.2}$$

where ρ_i, $i = 1,\dots,n$, are the car densities on incoming roads, while ρ_j, $j = n+1,\dots,n+m$, are the car densities on outgoing roads. In [12], the Riemann problem, that is the problem with constant initial data on each road, is solved maximizing a concave function of the fluxes and it is proved existence of weak solutions for Cauchy problems with suitable initial data of bounded variation. In this paper we assume that:

(A) there are some prescribed preferences of drivers, that is the traffic from incoming roads is distributed on outgoing roads according to fixed coefficients;

(B) respecting (A), drivers choose so as to maximize fluxes.

To deal with rule (A), we fix a traffic distribution matrix

$$A \doteq \{\alpha_{ji}\}_{j=n+1,\dots n+m,\, i=1,\dots,n} \in \mathbb{R}^{m\times n},$$

such that

$$0 < \alpha_{ji} < 1, \qquad \sum_{j=n+1}^{n+m} \alpha_{ji} = 1, \tag{1.3}$$

for each $i = 1, ..., n$ and $j = n+1, ..., n+m$, where α_{ji} is the percentage of drivers arriving from the $i-$th incoming road that take the $j-$th outgoing road. Notice that with only the rule (A) Riemann problems are still underdetermined. This choice represents a situation in which drivers have a final destination, hence distribute on outgoing roads according to a fixed law, but maximize the flux whenever possible. We are able to solve uniquely Riemann problems, under suitable conditions on the matrix A. Our main technique is the use of a front tracking algorithm and suitable approximations in order to control the total variation of the flux. We refer the reader to [6] for the general theory of conservation laws and for a discussion of wave front tracking algorithms.

The main difficulty in solving systems of conservation laws is the control of the total variation, see [6]. It is easy to see that for a single conservation law the total variation is decreasing, however in our case it may increase due to interaction of waves with junctions.

There is a natural lack of symmetry for big waves and bad data at junctions, since the role of entering roads is different from that of exiting ones. Similarly, for scalar conservation laws with discontinuous coefficients, one has to use a definition of strength for discontinuities of the coefficient, seen as waves, that is not symmetric but depends on the sign of the jump in the solution, see [13, 16, 17]. This is enough to control the total variation in that case, on the contrary our problem is more delicate. In fact, the variation can still increase due to interactions of waves with junctions (and there is no bound on the number of interactions). The conserved quantity is the total variation of the flux. We prove this fact for junctions with only two incoming roads and two outgoing ones. Unfortunately the total variation of the flux is not equivalent to the total variation of ρ, since $f'(\sigma) = 0$, and so it is not sufficient to prove existence of solutions. We need also some compactness arguments and some bound of big waves near the junctions.

Our techniques are quite flexible, so we can deal with time dependent coefficients for the rule (A). In particular, we can model traffic lights and also in this case the control of total variation is extremely delicate. An arbitrarily small change in the coefficients can produce waves whose strength is bounded away from zero. Still it is possible to consider periodic coefficients, a case of particular interest for applications. We can also deal with roads with different fluxes: this can be treated in the same way with the necessary notational modifications.

There is an interesting ongoing discussion on hydrodynamic modelization for heavy traffic flow. In particular some models using systems of two conservation laws have been proposed, see [4, 9, 11]. We do no treat this aspect.

The paper is organized as follows. In Section 2 we give the definition of weak entropic solution and following (A) and (B) we introduce an admissibility condition. In Section 3 we prove the existence and uniqueness of admissible solutions for the Riemann Problem in a junction, then using this we describe the construction of the approximants for the Cauchy Problem (see Section 4). In Section 5 we prove the monotonicity of the total variation of the flux and existence of admissible solutions for the Cauchy Problem with suitable BV initial data. In Section 6 we prove with a counterexample that the Lipschitz continuous dependence with respect to initial data does not hold, but we also show that this property holds under special assumptions. In Section 7 we describe what happens when there are traffic lights and time dependent coefficients. Finally, in Appendix B we show that the interaction of a small wave with a junction can produce a uniformly big wave.

2 Basic Definitions.

We consider a network of roads, that is modeled by a finite collection of intervals $I_i = [a_i, b_i] \subset \mathbb{R}$, $i = 1, \ldots, N$, $a_i < b_i$, possibly with either $a_i = -\infty$ or $b_i = +\infty$, on which we consider the equation (1.1). Hence the datum is given by a finite collection of functions ρ_i defined on $[0, +\infty[\times I_i$.

On each road I_i we want ρ_i to be a weak entropic solution, that is for every function $\varphi : [0, +\infty[\times I_i \to \mathbb{R}$ smooth with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \Big(\rho_i \frac{\partial \varphi}{\partial t} + f(\rho_i)\frac{\partial \varphi}{\partial x}\Big) dxdt = 0, \tag{2.4}$$

and for every $k \in \mathbb{R}$ and every $\tilde{\varphi} : [0, +\infty[\times I_i \to \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \Big(|\rho_i - k|\frac{\partial \tilde{\varphi}}{\partial t} + \text{sgn }(\rho_i - k)(f(\rho_i) - f(k))\frac{\partial \tilde{\varphi}}{\partial x}\Big) dxdt \geq 0. \tag{2.5}$$

It is well known that, for equation (1.1) on $\mathbb{R}$ and for every initial data in L^∞, there exists a unique weak entropic solution depending in a continuous way from the initial data in L^1_{loc}.

We assume that the roads are connected by some junctions. Each junction J is given by a finite number of incoming roads and a finite number of outgoing roads, thus we identify J with $((i_1, \ldots, i_n), (j_1, \ldots, j_m))$ where the first n–tuple indicates the set of incoming roads and the second m–tuple indicates the set of outgoing roads. We assume that each road can be incoming road at most for one junction and outgoing at most for one junction.

Hence the complete model is given by a couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I} = \{I_i : i = 1, \ldots, N\}$ is the collection of roads and $\mathcal{J}$ is the collection of junctions.

Fix a junction J with incoming roads, say $I_1, \ldots, I_n$, and outgoing roads, say $I_{n+1}, \ldots, I_{n+m}$. A weak solution at J is a collection of functions $\rho_l : [0, +\infty[\times I_l \to \mathbb{R}$, $l = 1, \ldots, n+m$, such that

$$\sum_{l=0}^{n+m}\left(\int_0^{+\infty}\int_{a_l}^{b_l}\left(\rho_l\frac{\partial\varphi_l}{\partial t}+f(\rho_l)\frac{\partial\varphi_l}{\partial x}\right)dxdt\right)=0, \tag{2.6}$$

for every φ_l, $l=1,\dots,n+m$ smooth having compact support in $]0,+\infty[\times]a_l,b_l]$ for $l=1,\dots,n$ (incoming roads) and in $]0,+\infty[\times[a_l,b_l[$ for $l=n+1,\dots,n+m$ (outgoing roads), that are also *smooth across the junction*, i.e.

$$\varphi_i(\cdot,b_i)=\varphi_j(\cdot,a_j),\qquad \frac{\partial\varphi_i}{\partial x}(\cdot,b_i)=\frac{\partial\varphi_j}{\partial x}(\cdot,a_j),\qquad i=1,...,n,\ j=n+1,...,n+m.$$

Remark 2.1 Let $\rho=(\rho_1,\dots,\rho_{n+m})$ be a weak solution at the junction such that each $x\to\rho_i(t,x)$ has bounded variation. We can deduce that ρ satisfies the *Rankine-Hugoniot Condition* at the junction J, namely

$$\sum_{i=1}^{n} f(\rho_i(t,b_i-))=\sum_{j=n+1}^{n+m} f(\rho_j(t,a_j+)), \tag{2.7}$$

for almost every $t>0$.

The rules (A) and (B) can be given explicitly only for solutions with bounded variation as in the next definition.

Definition 2.1 *Let $\rho=(\rho_1,\dots,\rho_{n+m})$ be such that $\rho_i(t,\cdot)$ is of bounded variation for every $t\geq 0$. Then ρ is an admissible weak solution of (1.1) related to the matrix A, satisfying (1.3), at the junction J if and only if the following properties hold:*

(i) ρ is a weak solution at the junction J;

(ii) $f(\rho_j(\cdot,a_j+))=\sum_{i=1}^{n}\alpha_{ji}f(\rho_i(\cdot,b_i-))$, for each $j=n+1,...,n+m$;

(iii) $\sum_{i=1}^{n} f(\rho_i(\cdot,b_i-))$ is maximum subject to (ii).

For every road $I_i=[a_i,b_i]$, if $a_i>-\infty$ and I_i is not the outgoing road of any junction, or $b_i<+\infty$ and I_i is not the incoming road of any junction, then a boundary data $\psi_i:[0,+\infty[\to\mathbb{R}$ is given. In this case we ask ρ_i to satisfy $\rho_i(t,a_i)=\psi_i(t)$ (or $\rho_i(t,b_i)=\psi_i(t)$) in the sense of [5]. The treatment of boundary data in the sense of [5] can be done in the same way as in [1, 2, 3], thus we treat the case without boundary data. All the stated results hold also for the case with boundary data with obvious modifications.

Our aim is to solve the Cauchy problem on $[0,+\infty[$ for a given initial and boundary data as in next definition.

Definition 2.2 *Given $\bar{\rho}_i : I_i \to \mathbb{R}$, $i = 1, \ldots, N$, L^∞ functions, a collection of functions $\rho = (\rho_1, \ldots, \rho_N)$, with $\rho_i : [0, +\infty[\times I_i \to \mathbb{R}$ continuous as functions from $[0, +\infty[$ into L^1_{loc}, is an admissible solution if ρ_i is a weak entropic solution to (1.1) on I_i, $\rho_i(0, x) = \bar{\rho}_i(x)$ a.e., and if at each junction ρ is a weak solution and is an admissible weak solution in case of bounded variation.*

On the flux f we make the following assumption

($\mathcal{F}$) $f : [0, 1] \to \mathbb{R}$ is smooth, strictly concave (i.e. $f'' \le -c < 0$ for some $c > 0$), $f(0) = f(1) = 0$. Therefore there exists a unique $\sigma \in]0, 1[$ such that $f'(\sigma) = 0$ (that is σ is a strict maximum).

3 The Riemann Problem.

In this section we study Riemann problems. For a scalar conservation law a Riemann problem is a Cauchy problem for an initial data of Heaviside type, that is piecewise constant with only one discontinuity. One looks for centered solutions, i.e. $\rho(t, x) = \phi(\frac{x}{t})$, which are the building blocks to construct solutions to the Cauchy problem via wave front tracking algorithm. These solutions are formed by continuous waves called rarefactions and by traveling discontinuities called shocks. The speed of waves are related to the values of f', see [6].

Analogously, we call Riemann problem for the road network the Cauchy problem corresponding to an initial data that is piecewise constant on each road. The solutions on each road I_i can be constructed in the same way as for the scalar conservation law, hence it suffices to describe the solution at junctions. Because of finite propagation speed, it is enough to study the Riemann Problem for a single junction.

Consider a junction J in which there are n roads with incoming traffic and m roads with outgoing traffic, and a traffic distribution matrix A. For simplicity we indicate by

$$(t, x) \in \mathbb{R}_+ \times I_i \mapsto \rho_i(t, x) \in [0, 1], \quad i = 1, ..., n, \tag{3.8}$$

the densities of the cars on the roads with incoming traffic and

$$(t, x) \in \mathbb{R}_+ \times I_j \mapsto \rho_j(t, x) \in [0, 1], \quad j = n + 1, ..., n + m \tag{3.9}$$

those on the roads with outgoing traffic, sse Figure 1.

We need some more notation:

Definition 3.1 *Let $\tau : [0, 1] \to [0, 1]$ be the map such that:*

1. *$f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0, 1]$;*

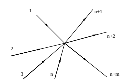

Figure 1: a junction.

2. $\tau(\rho) \neq \rho$ *for every* $\rho \in [0,1] \setminus \{\sigma\}$.

Clearly, τ is well defined and satisfies

$$0 \leq \rho \leq \sigma \Longleftrightarrow \sigma \leq \tau(\rho) \leq 1, \qquad \sigma \leq \rho \leq 1 \Longleftrightarrow 0 \leq \tau(\rho) \leq \sigma.$$

To state the main result of this section we need some assumption on the matrix A satisfied under generic conditions. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$ and for every subset $V \subset \mathbb{R}^n$ indicate by $V^\perp$ its orthogonal. Define for every $i = 1, \ldots, n$, $H_i = e_i^\perp$, i.e. the coordinate hyperplane orthogonal to e_i and for every $j = n+1, \ldots, n+m$ let $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn}) \in \mathbb{R}^n$ and define $H_j = \alpha_j^\perp$. Let $\mathcal{K}$ be the set of indices $k = (k_1, ..., k_\ell)$, $1 \leq \ell \leq n-1$, such that $0 \leq k_1 < k_2 < \cdots < k_\ell \leq n+m$ and for every $k \in \mathcal{K}$ set

$$H_k = \bigcap_{h=1}^{\ell} H_h.$$

Letting $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$, we assume

(C) for every $k \in \mathcal{K}$, $\mathbf{1} \notin H_k^\perp$.

Remark 3.1 Condition (C) is a technical condition, which allows us to have uniqueness to the maximization problem described in Theorem 3.1. From (C) we immediately derive $m \geq n$. Otherwise, since by definition $\mathbf{1} = \sum_{j=n+1}^{n+m} \alpha_j$, we get $\mathbf{1} \in H_k^\perp$, where

$$H_k = \cap_{j=n+1}^{n+m} H_j,$$

and it is clearly a contradiction. Moreover if $n \geq 2$, then (C) implies that, for every $j \in \{n+1, \ldots, n+m\}$ and for every distinct elements $i, i' \in \{1, \ldots, n\}$, it holds $\alpha_{ji} \neq \alpha_{ji'}$. Otherwise, without loss of generalities, we may suppose that $\alpha_{n+1,1} = \alpha_{n+1,2}$. If we consider

$$H_k = (\cap_{2<j\leq n} H_j) \cap H_{n+1},$$

then, by (C), there exists an element $(x_1, x_2, 0, \dots, 0) \in H$ such that $x_1 + x_2 \neq 0$ and $\alpha_{n+1,1}(x_1, x_2) = 0$.

In the case of a simple junction J with 2 incoming roads and 2 outgoing ones, the condition (C) is completely equivalent to the fact that, for every $j \in \{3, 4\}$, $\alpha_{j1} \neq \alpha_{j2}$.

Remark 3.2 Notice that the matrix A could have identical lines. For example the matrix

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \end{pmatrix}$$

satisfies the condition (C).

Theorem 3.1 *Consider a junction J, assume that the flux $f : [0,1] \to \mathbb{R}$ satisfies $(\mathcal{F})$ and the matrix A satisfies condition (C). For every $\rho_{1,0}, ..., \rho_{n+m,0} \in [0,1]$, there exists a unique admissible centered weak solution, in the sense of Definition 2.1, $\rho = \big(\rho_1, ..., \rho_{n+m}\big)$ of (1.1) at the junction J such that*

$$\rho_1(0, \cdot) \equiv \rho_{1,0}, ..., \rho_{n+m}(0, \cdot) \equiv \rho_{n+m,0}.$$

Moreover, there exists a unique $(n+m)-$tuple $(\hat{\rho}_1, ..., \hat{\rho}_{n+m}) \in [0,1]^{n+m}$ such that

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1], & \text{if } 0 \le \rho_{i,0} \le \sigma, \\ [\sigma, 1], & \text{if } \sigma \le \rho_{i,0} \le 1, \end{cases} \qquad i = 1, ..., n, \tag{3.10}$$

and

$$\hat{\rho}_j \in \begin{cases} [0, \sigma], & \text{if } 0 \le \rho_{j,0} \le \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[, & \text{if } \sigma \le \rho_{j,0} \le 1, \end{cases} \qquad j = n+1, ..., n+m. \tag{3.11}$$

Fixed $i \in \{1, ..., n\}$, if $\rho_{i,0} \le \hat{\rho}_i$, we have

$$\rho_i(t,x) = \begin{cases} \rho_{i,0}, & \text{if } x < \frac{f(\hat{\rho}_i) - f(\rho_{i,0})}{\hat{\rho}_i - \rho_{i,0}} t + b_i,\ t \ge 0, \\ \hat{\rho}_i, & \text{if } x > \frac{f(\hat{\rho}_i) - f(\rho_{i,0})}{\hat{\rho}_i - \rho_{i,0}} t + b_i,\ t \ge 0, \end{cases} \tag{3.12}$$

and, if $\hat{\rho}_i < \rho_{i,0}$,

$$\rho_i(t,x) = \begin{cases} \rho_{i,0}, & \text{if } x \le f'(\rho_{i,0})t + b_i,\ t \ge 0, \\ \big(f'\big)^{-1}\big((x - b_i)/t\big), & \text{if } f'(\rho_{i,0})t + b_i \le x \le f'(\hat{\rho}_i)t + b_i,\ t \ge 0, \\ \hat{\rho}_i, & \text{if } x > f'(\hat{\rho}_i)t + b_i,\ t \ge 0. \end{cases} \tag{3.13}$$

Fixed $j \in \{n+1, ..., n+m\}$, if $\rho_{j,0} \le \hat{\rho}_j$, we have

$$\rho_j(t,x) = \begin{cases} \hat{\rho}_j, & \text{if } x \le f'(\hat{\rho}_j)t + a_j,\ t \ge 0, \\ \big(f'\big)^{-1}\big((x - a_j)/t\big), & \text{if } f'(\hat{\rho}_j)t + a_j \le x \le f'(\rho_{j,0})t + a_j,\ t \ge 0, \\ \rho_{j,0}, & \text{if } x > f'(\rho_{j,0})t + a_j,\ t \ge 0, \end{cases} \tag{3.14}$$

and, if $\hat{\rho}_j < \rho_{j,0}$,

$$\rho_j(t,x) = \begin{cases} \hat{\rho}_j, & \text{if } x < \frac{f(\rho_{j,0})-f(\hat{\rho}_j)}{\rho_{j,0}-\hat{\rho}_j} t + a_j,\ t \geq 0, \\ \rho_{j,0}, & \text{if } x > \frac{f(\rho_{j,0})-f(\hat{\rho}_j)}{\rho_{j,0}-\hat{\rho}_j} t + a_j,\ t \geq 0. \end{cases} \tag{3.15}$$

Proof. Define the map

$$E : (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n \longmapsto \sum_{i=1}^{n} \gamma_i$$

and the sets

$$\Omega_i \doteq \begin{cases} [0, f(\rho_{i,0})], & \text{if } 0 \leq \rho_{i,0} \leq \sigma, \\ [0, f(\sigma)], & \text{if } \sigma \leq \rho_{i,0} \leq 1, \end{cases} \qquad i = 1, ..., n,$$

$$\Omega_j \doteq \begin{cases} [0, f(\sigma)], & \text{if } 0 \leq \rho_{j,0} \leq \sigma, \\ [0, f(\rho_{j,0})], & \text{if } \sigma \leq \rho_{j,0} \leq 1, \end{cases} \qquad j = n+1, ..., n+m,$$

$$\Omega \doteq \Big\{ (\gamma_1, ..., \gamma_n) \in \Omega_1 \times \ldots \times \Omega_n \Big| A \cdot (\gamma_1, ..., \gamma_n)^T \in \Omega_{n+1} \times \ldots \times \Omega_{n+m} \Big\}.$$

The set Ω is closed, convex and not empty. Moreover, by (C), ∇E is not orthogonal to any nontrivial subspace contained in a supporting hyperplane of Ω, hence there exists a unique vector $(\hat{\gamma}_1, ..., \hat{\gamma}_n) \in \Omega$ such that

$$E(\hat{\gamma}_1, ..., \hat{\gamma}_n) = \max_{(\gamma_1, \ldots, \gamma_n) \in \Omega} E(\gamma_1, ..., \gamma_n).$$

For every $i \in \{1, ..., n\}$, we choose $\hat{\rho}_i \in [0,1]$ such that

$$f(\hat{\rho}_i) = \hat{\gamma}_i, \quad \hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1], & \text{if } 0 \leq \rho_{i,0} \leq \sigma, \\ [\sigma, 1], & \text{if } \sigma \leq \rho_{i,0} \leq 1. \end{cases}$$

By $(\mathcal{F})$, $\hat{\rho}_i$ exists and is unique. Let

$$\hat{\gamma}_j \doteq \sum_{i=1}^{n} \alpha_{ji} \hat{\gamma}_i, \qquad j = n+1, ..., n+m,$$

and $\hat{\rho}_j \in [0,1]$ be such that

$$f(\hat{\rho}_j) = \hat{\gamma}_j, \quad \hat{\rho}_j \in \begin{cases} [0, \sigma], & \text{if } 0 \leq \rho_{j,0} \leq \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[, & \text{if } \sigma \leq \rho_{j,0} \leq 1. \end{cases}$$

Since $(\hat{\gamma}_1, ..., \hat{\gamma}_n) \in \Omega$, $\hat{\rho}_j$ exists and is unique for every $j \in \{n+1, \ldots, n+m\}$. Solving the Riemann Problem (see [6, Chapter 6]) on each road, the thesis is proved. □

4 The Wave Front Tracking Algorithm.

Once the solution to a Riemann problem is provided, we are able to construct piecewise constant approximations via wave-front tracking algorithm. The construction is very similar to that for scalar conservation law, see [6], hence we briefly describe it.

Let $\bar{\rho} = (\rho_1, \ldots, \rho_N)$ be a piecewise constant map defined on the road network. We want to construct a weak solution of (1.1) with initial condition $\rho(0, \cdot) \equiv \bar{\rho}$. We begin by solving the Riemann Problems on each road in correspondence of the jumps of $\bar{\rho}$ and the Riemann Problems at junctions determined by the values of $\bar{\rho}$ (see Theorem 3.1). We split each rarefaction wave into a rarefaction fan formed by rarefaction shocks, that are discontinuities traveling with the Rankine-Hugoniot speed. We always split rarefaction waves inserting the value σ (if it is in the range of the rarefaction). Moreover, we let any rarefaction shock with endpoint σ have velocity zero.

When a wave interacts with another one we simply solve the new Riemann Problem. Instead, when a wave reaches a junction, we solve the Riemann Problem at the junction. The number of waves may increase only for interactions of waves at junctions. Since the speeds of waves are bounded, there are finitely many waves on the network at each time $t \geq 0$. We call the obtained function *a wave front tracking approximate solution*. Given a general initial data, we approximate it by a sequence of piecewise constant functions and construct the corresponding approximate solutions. If they converge in L^1_{loc}, then the limit is a weak entropic solution on each road, see [6] for a proof.

5 Estimates on Flux Variation and Existence of Solutions.

This Section is devoted to the estimation of the total variation of the flux along a solution. From now on, we assume that every junction has exactly two incoming roads and two outgoing ones. The case where junctions have at most two incoming roads and at most two outgoing roads can be treated in the same way. So, for each junction J, the matrix A, defined in the introduction, takes the form

$$A = \begin{pmatrix} \alpha & \beta \\ 1-\alpha & 1-\beta \end{pmatrix}, \tag{5.16}$$

where $\alpha, \beta \in]0, 1[$ and $\alpha \neq \beta$, so that (C) is satisfied.

From now on we fix an approximate wave front tracking solution ρ, defined on the road network.

Definition 5.1 *For every road I_i, $i = 1, \dots, N$, we indicate by*

$$(\rho_-^\theta, \rho_+^\theta), \quad \theta \in \Theta = \Theta(\rho, t, i), \quad \Theta \text{ finite set,}$$

the discontinuities on road I_i at time t, and by $x^\theta(t)$, $\lambda^\theta(t)$, $\theta \in \Theta$, respectively their positions and velocities at time t. We also refer to the wave θ to indicate the discontinuity $(\rho_-^\theta, \rho_+^\theta)$.

For each discontinuity $(\rho_-^\theta, \rho_+^\theta)$ at time $\bar{t}$ on road I_i, we call $y^\theta(t)$, $t \in [\bar{t}, t_\theta]$, the trace of the wave so defined. We start with $y^\theta(\bar{t}) = x^\theta(\bar{t})$ and we continue up to the first interaction with another wave or a junction. If at time $\tilde{t}$ an interaction with a wave or a junction occurs, then either a single new wave $(\rho_-^{\tilde{\theta}}, \rho_+^{\tilde{\theta}})$ on road I_i is produced or no wave is produced. In the latter case we set $t_\theta = \tilde{t}$, otherwise we set $y^\theta(\tilde{t}) = x^{\tilde{\theta}}(\tilde{t})$ and follow $x^{\tilde{\theta}}(t)$ for $t \geq t^{\tilde{\theta}}$ up to next interaction and so on.

We start by proving some technical lemmata.

Lemma 5.1 *Fix a junction J and an incoming road I_i. Let θ be a wave on road I_i, produced at time $\bar{t}$ at J with a flux decrease, i.e. $x^\theta(\bar{t}) = b_i$, $\lambda^\theta(\bar{t}) < 0$ and $f(\rho_+^\theta) < f(\rho_-^\theta)$. Let y^θ be the traced wave and assume that there exists $\tilde{t}$, the first time of interaction of y^θ with J after $\bar{t}$. Then either y^θ interacts with another junction on $]\bar{t}, \tilde{t}[$ or, letting $\theta_1, \dots, \theta_l$ be the waves interacting with y^θ at times $t_m \in]\bar{t}, \tilde{t}[$, $m = 1, \dots, l$, ($t_1 < t_2 < \dots < t_l$), we have:*

$$\begin{aligned} & \left| f(\rho(\tilde{t} - \varepsilon, y^\theta(\tilde{t} - \varepsilon)+)) - f(\rho(\tilde{t} - \varepsilon, y^\theta(\tilde{t} - \varepsilon)-)) \right| \\ \leq & \sum_{m=1}^{l} \left| f(\rho(t_m - \varepsilon, x^{\theta_m}(t_m - \varepsilon)+)) - f(\rho(t_m - \varepsilon, x^{\theta_m}(t_m - \varepsilon)-)) \right| - \left| f(\rho_-^\theta) - f(\rho_+^\theta) \right|, \end{aligned}$$

for $\varepsilon > 0$ small enough. This means that the initial flux variation along y^θ is canceled. The same conclusion holds for an outcoming road I_j.

Proof. Consider the wave $(\rho_-^\theta, \rho_+^\theta)$ as in the statement, then it is a shock with negative velocity and $\rho_+^\theta > \max\{\rho_-^\theta, \tau(\rho_-^\theta)\}$. If y^θ interacts with another junction, then there is nothing to prove. So, we assume that y^θ does not interact with another junction. At time t_1, the wave θ_1 interacts with y^θ. We analyze first the case of interaction from the left of y^θ. We have two possibilities:

1. $\rho_-^{\theta_1} \in [0, \tau(\rho_+^\theta)]$. In this case we have total cancellation of the flux variation and so

$$\left| f(\rho_+^\theta) - f(\rho_-^{\theta_1}) \right| = \left| f(\rho_-^{\theta_1}) - f(\rho_-^\theta) \right| - \left| f(\rho_-^\theta) - f(\rho_+^\theta) \right|.$$

Therefore the thesis easily follows.

2. $\rho_-^{\theta_1} \in]\tau(\rho_+^\theta), \rho_+^\theta]$. In this case the wave y^θ after the time interaction t_1 has the same nature of y^θ before t_1, i.e.

$$\max\{\rho(t_1+, y^\theta(t_1+)-), \tau(\rho(t_1+, y^\theta(t_1+)-))\} < \rho(t_1+, y^\theta(t_1+)+).$$

We consider now the case of interaction from the right of y^θ. It is clear that $\rho_+^{\theta_1} \in]\rho_-^\theta, 1]$. If moreover $f(\rho_+^{\theta_1}) \geq f(\rho_-^\theta)$, then we have total cancellation of the flux and we conclude as before. If instead $f(\rho_+^{\theta_1}) < f(\rho_-^\theta)$, then the wave y^θ after the time t_1 has the same nature of y^θ before t_1.

We repeat this argument at each interaction time t_m. If at some t_m we have total cancellation of the flux, then we conclude. Therefore we may suppose that at each t_m total cancellation of the flux does not occur. Since the nature of the wave y^θ does not change, we have

$$\max\{\rho(\tilde{t}-, y^\theta(\tilde{t}-)-), \tau(\rho(\tilde{t}-, y^\theta(\tilde{t}-)-))\} < \rho(\tilde{t}-, y^\theta(\tilde{t}-)+),$$

and hence the speed $\lambda^\theta(\tilde{t}-)$ is negative, which contradicts the fact that y^θ interacts with J at time $\tilde{t}$. □

Lemma 5.2 *Fix a junction J and an incoming road I_i. Let θ be a wave on road I_i, produced at time $\bar{t}$ at J by a flux increase, i.e. $x^\theta(\bar{t}) = b_i$, $\lambda^\theta(\bar{t}) < 0$ and $f(\rho_+^\theta) > f(\rho_-^\theta)$. Let y^θ be the traced wave and assume that there exists $\tilde{t}$, the first time of interaction of y^θ with J after $\bar{t}$ and that y^θ does not interact with other junctions in $]\bar{t}, \tilde{t}[$. If y^θ does not cancel the flux variation, then it produces a flux decrease at J at $\tilde{t}$, i.e.*

$$f(\rho(\tilde{t}-\varepsilon, y^\theta(\tilde{t}-\varepsilon)-)) < f(\rho(\tilde{t}-\varepsilon, y^\theta(\tilde{t}-\varepsilon)+)),$$

for $\varepsilon > 0$ small enough. The same holds for outgoing roads.

Proof. Since $\lambda^\theta(\bar{t}) < 0$ and $f(\rho_+^\theta) > f(\rho_-^\theta)$, then $\rho_-^\theta > \sigma$. Moreover the wave $(\rho_-^\theta, \rho_+^\theta)$ is a rarefaction fan, hence $\sigma < \rho_+^\theta < \rho_-^\theta$.

If an interaction on the right with a wave θ_1 happens, then $\rho_+^{\theta_1} \in]\rho_-^\theta, 1]$ and we have total cancellation of the flux variation. Therefore we may suppose that an interaction on the left with a wave θ_1 happens. In this case we have two possibilities:

1. $\rho_-^{\theta_1} \in [0, \tau(\rho_+^\theta)[$;
2. $\rho_-^{\theta_1} \in [\tau(\rho_+^\theta), \rho_+^\theta[$.

In the latter case we have total cancellation of the flux variation and so we conclude. In the first case, instead, the type of the wave changes, since

$$0 < \rho_-^{\theta_1} < \tau(\rho_+^\theta) \leq \sigma \leq \rho_+^\theta < 1.$$

The speed of the wave y^θ after this interaction is positive and if there are no more interaction, then we have the thesis since $f(\rho_-^{\theta_1}) < f(\rho_+^{\theta})$. Thus we suppose that an interaction with a wave θ_2 happens. If it is an interaction from the left, then the possibilities are the followings:

1. $\rho_-^{\theta_2} \in [0, \tau(\rho_+^\theta)[$. We do not have total cancellation of the flux variation, but the type of the wave does not change and the situation is identical to the previous one.
2. $\rho_-^{\theta_2} \in [\tau(\rho_+^\theta), \sigma[$. We have total cancellation of the flux variation and so we conclude.

If it is an interaction from the right, then the possibilities are the followings:

1. $\rho_+^{\theta_2} \in [\sigma, \tau(\rho_-^{\theta_1})[$. We do not have total cancellation of the flux variation, but the type of the wave does not change.
2. $\rho_+^{\theta_2} \in [\tau(\rho_-^{\theta_1}), 1]$. We have total cancellation of the flux variation and so we conclude.

The conclusion now easily follows repeating this argument. If at each interaction we do not have total cancellation of the flux variation, then we necessary have that

$$f(\rho(\tilde{t} - \varepsilon, y^\theta(\tilde{t} - \varepsilon)-)) < f(\rho(\tilde{t} - \varepsilon, y^\theta(\tilde{t} - \varepsilon)+)),$$

for $\varepsilon > 0$ small enough, which concludes the proof. □

Lemma 5.3 *Fix a junction J. If a wave interacts with the junction J from an incoming road at time $\bar{t}$, then*

$$\mathit{Tot.Var.}\big(f(\rho(\bar{t}+, \cdot))\big) = \mathit{Tot.Var.}\big(f(\rho(\bar{t}-, \cdot))\big). \tag{5.17}$$

PROOF. For simplicity let us assume that I_1, I_2 are the incoming roads and I_3, I_4 are the outgoing ones. Let $(\rho_{1,0}, ..., \rho_{4,0})$ be an equilibrium configuration at the junction J. We assume that the wave is coming from the first road and that it is given by the values $(\rho_1, \rho_{1,0})$. Let us define the incoming flux

$$f^{in}(y) \doteq \begin{cases} f(y), & \text{if } 0 \le y \le \sigma, \\ f(\sigma), & \text{if } \sigma \le y \le 1, \end{cases} \tag{5.18}$$

and the outgoing flux

$$f^{out}(y) \doteq \begin{cases} f(\sigma), & \text{if } 0 \le y \le \sigma, \\ f(y), & \text{if } \sigma \le y \le 1. \end{cases} \tag{5.19}$$

Clearly, since the wave on the first road has positive velocity, we have

$$0 \le \rho_1 < \sigma. \tag{5.20}$$

Let $(\hat\rho_1, ..., \hat\rho_4)$ be the solution of the Riemann Problem in the junction J with initial data $(\rho_1, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ (see Theorem 3.1). By definition, $\big(f(\rho_{1,0}), f(\rho_{2,0})\big)$ is the maximum point of the map E on the domain

$$\Omega_0 \doteq \Big\{(\gamma_1, \gamma_2) \in \Omega_{1,0} \times \Omega_{2,0} \big| A \cdot (\gamma_1, \gamma_2)^T \in \Omega_{3,0} \times \Omega_{4,0}\Big\},$$

and $\big(f(\hat\rho_1), f(\hat\rho_2)\big)$ is the maximum point of the map E on the domain

$$\hat\Omega \doteq \Big\{(\gamma_1, \gamma_2) \in \Omega_1 \times \Omega_{2,0} \big| A \cdot (\gamma_1, \gamma_2)^T \in \Omega_{3,0} \times \Omega_{4,0}\Big\},$$

where

$$\Omega_{j,0} \doteq \begin{cases} [0, f^{in}(\rho_{j,0})], & \text{if } j = 1, 2, \\ [0, f^{out}(\rho_{j,0})], & \text{if } j = 3, 4, \end{cases}$$

and, by (5.20),

$$\Omega_1 \doteq [0, f^{in}(\rho_1)] = [0, f(\rho_1)].$$

It is also clear that

$$\big(f(\rho_{1,0}), f(\rho_{2,0})\big) \in \partial\Omega_0, \qquad \big(f(\hat\rho_1), f(\hat\rho_2)\big) \in \partial\hat\Omega.$$

For simplicity we use the notation (5.16).

We distinguish two cases. First we suppose that

$$f(\rho_1) < f(\rho_{1,0}), \tag{5.21}$$

(equality can not happen in the previous equation because the wave would have velocity zero). Then $\hat\Omega \subset \Omega_0$ and

$$f(\hat\rho_1) \le f(\rho_1), \qquad f(\hat\rho_1) + f(\hat\rho_2) \le f(\rho_{1,0}) + f(\rho_{2,0}). \tag{5.22}$$

We claim that

$$f(\rho_{2,0}) \le f(\hat\rho_2), \quad f(\hat\rho_3) \le f(\rho_{3,0}), \quad f(\hat\rho_4) \le f(\rho_{4,0}). \tag{5.23}$$

The points $\big(f(\rho_{1,0}), f(\rho_{2,0})\big)$, $\big(f(\hat\rho_1), f(\hat\rho_2)\big)$ are on the boundaries of Ω_0, $\hat\Omega$ respectively, where the function E attains the maximum, hence each one is at least on one of the curves

$$\alpha\gamma_1 + \beta\gamma_2 = f^{out}(\rho_{3,0}), \qquad (1-\alpha)\gamma_1 + (1-\beta)\gamma_2 = f^{out}(\rho_{4,0}), \qquad \gamma_2 = f^{in}(\rho_{2,0}).$$

Let us assume that the two points are on the same curve, the other cases being similar,

$$\alpha\gamma_1 + \beta\gamma_2 = f^{out}(\rho_{3,0}). \tag{5.24}$$

Observe that the map E is increasing on the curve

$$\gamma_1 \mapsto \Big(\gamma_1, \frac{f^{out}(\rho_{3,0})}{\beta} - \frac{\alpha}{\beta}\gamma_1\Big),$$

otherwise we contradict the maximality of E at $\big(f(\rho_{1,0}), f(\rho_{2,0})\big)$. Thus $\alpha < \beta$, $\hat\rho_1 = \rho_1$, the first two inequalities in (5.23) hold and

$$f(\hat\rho_1) = f(\rho_1), \qquad f(\hat\rho_2) > f(\rho_{2,0}), \qquad f(\hat\rho_3) = f(\rho_{3,0}) = f^{out}(\rho_{3,0}). \tag{5.25}$$

On the other hand, by (5.22), we have

$$\begin{aligned} f(\hat\rho_4) &= (1-\alpha)f(\hat\rho_1) + (1-\beta)f(\hat\rho_2) \leq \\ &\leq (1-\alpha)\big(f(\rho_{1,0}) + f(\rho_{2,0}) - f(\hat\rho_2)\big) + (1-\beta)f(\hat\rho_2) = \\ &= (1-\alpha)\big(f(\rho_{1,0}) + f(\rho_{2,0})\big) + \big(\alpha-\beta\big)f(\hat\rho_2) \leq \\ &\leq (1-\alpha)\big(f(\rho_{1,0}) + f(\rho_{2,0})\big) + \big(\alpha-\beta\big)f(\rho_{2,0}) = f(\rho_{4,0}). \end{aligned}$$

Thus (5.23) holds. Using the Rankine–Hugoniot Condition (2.7) at the junction J, and using (5.23), and (5.25), we get

$$\begin{aligned} &\text{Tot.Var.}\big(f(\rho(\bar t+,\cdot))\big) = \\ &\quad = |f(\hat\rho_1) - f(\rho_1)| + |f(\hat\rho_2) - f(\rho_{2,0})| + |f(\hat\rho_3) - f(\rho_{3,0})| + |f(\hat\rho_4) - f(\rho_{4,0})| = \\ &\quad = \big(f(\hat\rho_2) - f(\rho_{2,0})\big) + \big(f(\rho_{3,0}) - f(\hat\rho_3)\big) + \big(f(\rho_{4,0}) - f(\hat\rho_4)\big) = \\ &\quad = f(\rho_{1,0}) - f(\hat\rho_1) = f(\rho_{1,0}) - f(\rho_1) = \text{Tot.Var.}\big(f(\rho(\bar t-,\cdot))\big). \end{aligned}$$

Suppose now that

$$f(\rho_{1,0}) < f(\rho_1),$$

then $\rho_{1,0} < \rho_1 < \sigma$ and $\Omega_0 \subset \hat\Omega$. Assuming again that both points of maximum of the function E are on the curve (5.24), we have

$$f(\hat\rho_1) = f(\rho_1), \quad f(\hat\rho_2) \leq f(\rho_{2,0}), \quad f(\rho_{3,0}) = f(\hat\rho_3), \quad f(\rho_{4,0}) \leq f(\hat\rho_4).$$

By the Rankine Hugoniot Condition at the junction J (see (2.7)), we have

$$\begin{aligned} &\text{Tot.Var.}\big(f(\rho(\bar t+,\cdot))\big) = \\ &\quad = |f(\hat\rho_1) - f(\rho_1)| + |f(\hat\rho_2) - f(\rho_{2,0})| + |f(\hat\rho_3) - f(\rho_{3,0})| + |f(\hat\rho_4) - f(\rho_{4,0})| = \\ &\quad = \big(f(\rho_{2,0}) - f(\hat\rho_2)\big) + \big(f(\hat\rho_3) - f(\rho_{3,0})\big) + \big(f(\hat\rho_4) - f(\rho_{4,0}))\big) = \\ &\quad = f(\hat\rho_1) - f(\rho_{1,0}) = f(\rho_1) - f(\rho_{1,0}) = \text{Tot.Var.}\big(f(\rho(\bar t-,\cdot))\big). \end{aligned}$$

This concludes the proof. □

Lemma 5.4 *Consider a network $(\mathcal{I}, \mathcal{J})$. We have*

$$\textit{Tot.Var.}\big(f(\rho(0+,\cdot))\big) \leq \textit{Tot.Var.}\big(f(\rho(0,\cdot))\big) + 2Rf(\sigma),$$

where R is the total number of roads of the network.

Proof. At time $t = 0$ we can have an instantaneous increase of the total variation of the flux due to the waves generated by the Riemann problems in the junctions. Clearly, this increase can be estimated by the maximum number of waves generated in the junctions ($\leq 2R$) times the maximum variation of the flux on each road ($\leq f(\sigma)$). □

We are now ready to prove the following.

Lemma 5.5 *Consider a road network $(\mathcal{I}, \mathcal{J})$. For some $K > 0$, we have*

$$\begin{aligned} \textit{Tot.Var.}\big(f(\rho(t+,\cdot))\big) &\leq e^{Kt}\,\textit{Tot.Var.}\big(f(\rho(0+,\cdot))\big) \leq \\ &\leq e^{Kt}\big(\textit{Tot.Var.}\big(f(\rho(0,\cdot))\big) + 2Rf(\sigma)\big), \end{aligned}$$

for each $t \geq 0$.

Proof. Fix a junction J. Notice that there exists a constant C_J, depending on the coefficients of the matrix A at J, so that each interaction of a wave with J provokes an increase of flux variation at most by a factor C_J. More precisely, if $\text{Tot.Var.}_f^{\pm}$ is the flux variation of waves before and after the interaction then $\text{Tot.Var.}_f^{+} \leq C_J \text{Tot.Var.}_f^{-}$.

Consider a wave θ interacting with the junction J, then from Lemma 5.3 the flux variation can increase only if the wave is coming from an outgoing road. Let $\theta_1, \ldots, \theta_4$ be the waves so produced. Thanks to Lemma 5.1 waves produced by a flux decrease can not interact with the junction J without canceling the flux variation or reaching another junction. Moreover, by Lemma 5.2, every θ_i can come back to the junction J (without interacting with other junctions) only with a decrease of the flux. Now notice that a wave with decreasing flux interacting with J always produces a flux decrease on outgoing roads. Hence, waves θ_i may come back to the junction only with decreasing flux, thus, by Lemma 5.1, producing other waves that can not come back to the junction, unless they cancel their flux variation or interact with other junctions. Finally, each wave flux variation can be magnified just twice by a factor C_J interacting only with junction J and not with other junctions.

Now let δ be the minimum length of a road, i.e. $\delta = \min_{i\in\mathcal{I}}(b_i - a_i)$, and $\hat{\lambda}$ be the maximum speed of a wave, i.e. $\hat{\lambda} = \max\{f'(0), |f'(1)|\}$. Then each wave takes at least time $\delta/\hat{\lambda}$ to go from one junction to another.

Finally, recalling that the total variation of the flux may only decrease for interactions on roads, we get that a magnification of flux variation of a factor $C_{\mathcal{J}} = \max_{J\in\mathcal{J}} C_J^2$ may occur only once on each time interval of length $\delta/\hat{\lambda}$. We thus get:

$$\begin{aligned} \text{Tot.Var.}\big(f(\rho(t+,\cdot))\big) &\leq C_{\mathcal{J}}^{\frac{t\hat{\lambda}}{\delta}}\,\text{Tot.Var.}\big(f(\rho(0+,\cdot))\big) = \\ &= e^{Kt}\text{Tot.Var.}\big(f(\rho(0+,\cdot))\big), \end{aligned}$$

where $K = \hat{\lambda}\log(C_{\mathcal{J}})/\delta$. □

Definition 5.2 *Consider a road network $(\mathcal{I}, \mathcal{J})$ and an approximate wave front tracking solution ρ. For every road I_i, we define two curves $Y_-^{i,\rho}(t)$, $Y_+^{i,\rho}(t)$, called* Boundary of Extremal Flux*, briefly BEF, in the following way. We set the initial condition $Y_-^{i,\rho}(0) = a_i$, $Y_+^{i,\rho}(0) = b_i$ (if $a_i = -\infty$, then $Y_-^{i,\rho} \equiv -\infty$ and if $b_i = +\infty$, then $Y_+^{i,\rho} \equiv +\infty$). We let $Y_\pm^{i,\rho}(t)$ follow the generalized characteristic as defined in [10], as long as they lie inside $]a_i, b_i[$, otherwise we set $Y_-^{i,\rho}(t) = a_i$, $Y_+^{i,\rho}(t) = b_i$. Finally, let $\bar{t}$ be the first time $\bar{t}$ such that $Y_-^{i,\rho}(\bar{t}) = Y_+^{i,\rho}(\bar{t})$ (possibly $\bar{t} = +\infty$), then we let $Y_\pm^{i,\rho}$ defined on $[0, \bar{t}]$. We define the sets*

$$D_1^i(\rho) = \left\{(t, x) : t \in [0, \bar{t}) : Y_-^{i,\rho}(t) \le x \le Y_+^{i,\rho}(t)\right\},$$

and

$$D_2^i(\rho) = [0, +\infty) \times [a_i, b_i] \setminus D_1^i(\rho).$$

Clearly $Y_\pm^i(t)$ bound the set on which the datum is not influenced by the other roads through the junctions.

Definition 5.3 *Fix an approximate wave front tracking solution ρ and a road I_i, $i = 1, \dots, N$. A wave θ in I_i is said a big wave if*

$$sgn(\rho_-^\theta - \sigma) \cdot sgn(\rho_+^\theta - \sigma) \le 0.$$

Definition 5.4 *Fix an approximate wave front tracking solution ρ and a junction J. We say that an incoming road I_i has a bad data at J at time $t > 0$ if*

$$\rho_i(t, b_i-) \in [0, \sigma].$$

We say that an outgoing road I_j has a bad data at J at time $t > 0$ if

$$\rho_j(t, a_j+) \in [\sigma, 1].$$

Lemma 5.6 *For every $t \ge 0$, there exist at most two big waves on*

$$\left\{x : (t, x) \in D_2^i(\rho)\right\} \subseteq [a_i, b_i].$$

Proof. A big wave can be generated at a junction J on road I_i, say from a_i, only if there is a bad data at J in a_i. No other big waves can enter the road I_i from a_i, unless a big wave is absorbed by the junction J. Then we reach the conclusion. □

Theorem 5.1 *Fix a road network $(\mathcal{I}, \mathcal{J})$. For every $C > 0$ and for every $T > 0$ there exists an admissible solution defined on $[0, T]$ for every initial data $\bar{\rho} \in cl\{\rho : TV(\rho) \le C\}$, where cl indicates the closure in L^1.*

Proof. We fix a sequence of initial data $\bar{\rho}_\nu$ piecewise constant such that $TV(\bar{\rho}_\nu) \le C$ for every $\nu \ge 0$ and $\bar{\rho}_\nu \to \bar{\rho}$ in L^1 as $\nu \to +\infty$. For each $\bar{\rho}_\nu$ we consider an approximate wave front tracking solution ρ_ν such that $\rho_\nu(0, x) = \bar{\rho}_\nu(x)$.

For every road I_i, we notice that on $D_1^i(\rho_\nu)$, ρ_ν is not influenced by other roads and so the estimates of [6] hold. Since the curves $Y_\pm^{i,\rho_\nu}$ are Lipschitz continuous, it is clear that they converge to a limit curve and hence the regions $D_1^i(\rho_\nu)$ converge to a limit region D_1^i. Then $\rho_\nu \to \rho$ on D_1^i with ρ admissible solution to the Cauchy problem.

On $D_2^i := [0, +\infty[\times [a_i, b_i] \setminus D_1^i$, we have that, up to a subsequence, $\rho_\nu \rightharpoonup^* \rho$ weak* on L^1 and, by [6, Theorem 2.4], $f(\rho_\nu) \to \bar{f}$ in L^1 for some $\bar{f}$. By Lemma 5.6, there are at most two big waves, hence, splitting the domain D_2^i in a finite number of pieces where we can invert the function f, we have that $\rho_\nu \to f^{-1}(\bar{f})$ in L^1. Together with $\rho_\nu \rightharpoonup^* \rho$ weak* on L^1, we conclude that $\rho_\nu \to \rho$ strongly in L^1.

The other requirements of the definition of admissible solution are clearly satisfied. □

6 Lipschitz continuous dependence: a counterexample.

In this section we assume that every junction has exactly two incoming roads and two outgoing ones. For every junctions we follow the notation (5.16). We present a counterexample to the Lipschitz continuous dependence by initial data with respect to the L^1–norm. The continuous dependence by initial data with respect the L^1–norm remains an open problem. The counterexample is constructed using shifts of waves as in the spirit of [7], to which we refer the reader for general theory.

We show that, for every $C > 0$, it is possible to choose two piecewise constant initial data, which are exactly the same except for a shift ξ of a discontinuity, such that the L^1–distance of the two corresponding solutions increases by the multiplicative factor C. Obviously, the L^1–distance of the initial data is finite and given by $|\xi\, \Delta\rho|$, where ξ is the shift and $\Delta\rho$ is the jump across the corresponding discontinuity. From now on, we consider a junction J, satisfying condition (C), with I_1, I_2 as incoming roads and I_3, I_4 as outgoing ones. Moreover we suppose that the entries of the matrix A satisfy $\alpha < \beta$.

First we need some technical lemmas. The first one is well–known; we report the proof for reader's convenience.

Lemma 6.1 *Let us consider in a road two wave fronts with speeds λ_1 and λ_2 respectively. At a certain time $\bar{t}$ they interact together producing a wave front with speed λ_3. If the first wave is shifted by ξ_1 and the second wave by ξ_2, then the shift of the resulting wave is given*

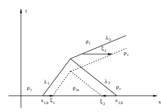

Figure 2: Shifts of waves.

by

$$\xi_3 = \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_2}\xi_1 + \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}\xi_2. \tag{6.26}$$

Moreover we have that

$$\Delta\rho_3\,\xi_3 = \Delta\rho_1\,\xi_1 + \Delta\rho_2\,\xi_2, \tag{6.27}$$

where $\Delta\rho_i$ are the signed strengths of the corresponding waves.

Proof. We suppose that ρ_l and ρ_m are the left and the right values of the wave with speed λ_1 and ρ_m and ρ_r are the left and the right values of the wave with speed λ_2, see Figure 2.

So $\Delta\rho_1 = \rho_m - \rho_l$, $\Delta\rho_2 = \rho_r - \rho_m$ and $\Delta\rho_3 = \rho_r - \rho_l$. The two wave fronts have respectively equation

$$x = \lambda_1 t + x_{1,0}, \qquad x = \lambda_2 t + x_{2,0},$$

where $x_{1,0}$ and $x_{2,0}$ are the initial positions of the wave fronts with speed λ_1 and λ_2 respectively. Therefore they interact at the point

$$(\bar{x}, \bar{t}) = \left(\lambda_1 \frac{x_{1,0} - x_{2,0}}{\lambda_2 - \lambda_1} + x_{1,0}, \frac{x_{1,0} - x_{2,0}}{\lambda_2 - \lambda_1}\right).$$

If we consider the shifts, then the two wave fronts interact at the point

$$(\tilde{x}, \tilde{t}) = \left(x_{1,0} + \xi_1 + \lambda_1 \frac{(x_{2,0} + \xi_2) - (x_{1,0} + \xi_1)}{\lambda_1 - \lambda_2}, \frac{(x_{2,0} + \xi_2) - (x_{1,0} + \xi_1)}{\lambda_1 - \lambda_2}\right),$$

and consequently

$$\xi_3 = \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_2}\xi_1 + \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}\xi_2.$$

Multiplying equation (6.26) by $\Delta\rho_3 = \Delta\rho_1 + \Delta\rho_2$, we easily deduce (6.27). □

Lemma 6.2 *Let us consider a junction J with the incoming roads I_1 and I_2 and the outgoing roads I_3 and I_4. Let us suppose that at a certain time a wave in a road I_i ($i \in \{1, \ldots, 4\}$)*

interacts with J without producing waves in the same road I_i. If ξ_i is the shift of the wave in I_i, then the shift ξ_j produced in a different road I_j ($j \in \{1, \ldots, 4\} \setminus \{i\}$) satisfies:

$$\xi_j \left(\rho_j^+ - \rho_j^-\right) = \frac{\Delta\gamma_j}{\Delta\gamma_i} \xi_i \left(\rho_i^+ - \rho_i^-\right), \tag{6.28}$$

where $\Delta\gamma_l$ ($l \in \{i, j\}$) represents the variation of the flux in the road I_l and ρ_l^-, ρ_l^+ ($l \in \{i, j\}$) are the states at J in the road I_l respectively before and after the interaction.

Proof. For simplicity let us consider the case $i = 1$ and $j = 3$, the other cases being completely similar. Applying the shift ξ_1 to the wave (ρ_1^+, ρ_1^-), the interaction of the wave with J is shifted in time by

$$-\xi_1 \frac{\rho_1^+ - \rho_1^-}{f(\rho_1^+) - f(\rho_1^-)} = -\xi_1 \frac{\rho_1^+ - \rho_1^-}{\Delta\gamma_1}.$$

The shift in time in I_3 must be the same and so

$$\xi_1 \frac{\rho_1^+ - \rho_1^-}{\Delta\gamma_1} = \xi_3 \frac{\rho_3^+ - \rho_3^-}{\Delta\gamma_3},$$

which concludes the lemma. □

Remark 6.1 It is easy to understand that the coefficient of multiplication $\Delta\gamma_j / \Delta\gamma_i$ in the previous lemma depends by the entries of the matrix A. For example, under the same hypotheses of the previous lemma, if a wave in the I_1 road interacts with J producing a variation of the flux $\Delta\gamma_1$ and if no wave is produced in I_1 and I_2, then

$$\Delta\gamma_3 = \alpha\Delta\gamma_1, \qquad \Delta\gamma_4 = (1 - \alpha)\Delta\gamma_1.$$

Consequently in this case

$$\frac{\Delta\gamma_3}{\Delta\gamma_1} = \alpha, \quad \frac{\Delta\gamma_4}{\Delta\gamma_1} = 1 - \alpha.$$

The other cases are similar.

The following lemma is the first step in order to show that the Lipschitz dependence by initial data does not hold in our setting. More precisely, we show that there exists a simple configuration of waves and of shifts, which, after interactions with J, produces an increase of the L^1-distance.

Lemma 6.3 *There exists an equilibrium configuration $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ in J, data $\bar{\rho}_2$, $\bar{\rho}_3$, ρ_3^* and shift $\xi_{3,0}$ such that, if we start from the equilibrium configuration at J, then the followings happen in chronological order:*

1. *the initial distance in* L^1 *is* $\xi_{3,0}\,|\rho_{3,0}-\rho_3^*|$*;*

2. *the wave* $(\rho_{3,0},\rho_3^*)$ *in* I_3 *with shift* $\xi_{3,0}$ *interacts with* J*;*

3. *waves are produced only in* I_2 *and* I_4*;*

4. *the wave on road* I_2 *interacts with* $(\bar\rho_2,\rho_{2,0})$ *producing a new wave;*

5. *the new wave from road* I_2 *interacts with* J*;*

6. *waves are produced only in* I_3 *and* I_4*;*

7. *in* I_4 *the* L^1*–distance after the interactions, is equal to*
$$2\frac{1-\beta}{\beta}\left|\xi_{3,0}\left(\rho_3^*-\rho_{3,0}\right)\right|,$$
and the L^1*–distance on road* I_3 *is equal to* $\xi_{3,0}\,|\rho_{3,0}-\rho_3^*|$*.*

PROOF. Let $(\rho_{1,0},\rho_{2,0},\rho_{3,0},\rho_{4,0})$ be an equilibrium configuration in J such that
$$0<\rho_{1,0}<\sigma,\quad 0<\rho_{2,0}<\sigma,\quad 0<\rho_{3,0}<\sigma,\quad 0<\rho_{4,0}<\sigma.$$
In road I_3, we consider a wave with negative speed $(\rho_{3,0},\rho_3^*)$ with shift $\xi_{3,0}$. Since $(\rho_{3,0},\rho_3^*)$ has negative speed, then $\rho_3^*>\tau(\rho_{3,0})$. Initially the L^1–distance of the two solutions is given by $|\xi_{3,0}(\rho_{3,0}-\rho_3^*)|$. When this wave interacts with J, new waves are produced in I_2 and I_4. It is possible, since $\alpha<\beta$. Therefore the new solution to the Riemann Problem at J is given by
$$\left(\rho_{1,0},\hat\rho_2,\hat\rho_3,\hat\rho_4\right),$$
where $\tau(\rho_{2,0})<\hat\rho_2<1$, $0<\hat\rho_4<\rho_{4,0}$. Moreover some shifts $\hat\xi_2$ and $\hat\xi_4$ are produced in roads I_2 and I_4 respectively, where obviously $\hat\xi_2$ has the same sign of $\xi_{3,0}$ while $\hat\xi_4$ has opposite sign. By Lemma 6.2, we have
$$\begin{cases}\hat\xi_2(\hat\rho_2-\rho_{2,0})=\frac{1}{\beta}\xi_{3,0}(\rho_3^*-\rho_{3,0}),\\ \hat\xi_4(\hat\rho_4-\rho_{4,0})=\frac{1-\beta}{\beta}\xi_{3,0}(\rho_3^*-\rho_{3,0}).\end{cases}$$
Then we may suppose that the wave $(\bar\rho_2,\rho_{2,0})$ in the road I_2 with shift $\bar\xi_2=0$ interacts with the wave $(\rho_{2,0},\hat\rho_2)$ producing a wave $(\bar\rho_2,\hat\rho_2)$ with positive speed and with shift $\tilde\xi_2$. This happens if $0<\bar\rho_2<\tau(\hat\rho_2)$ and in this case:
$$\tilde\xi_2(\hat\rho_2-\bar\rho_2)=\hat\xi_2(\hat\rho_2-\rho_{2,0})=\frac{1}{\beta}\xi_{3,0}(\rho_3^*-\rho_{3,0}).$$
Then, after the interaction of the wave $(\bar\rho_2,\hat\rho_2)$ with J, the new solution of the Riemann Problem at J is given by
$$\left(\rho_{1,0},\bar\rho_2,\hat\rho_3,\bar\rho_4\right),$$

where $0 < \hat{\rho}_3 < \tau(\rho_3^*)$ and $0 < \bar{\rho}_4 < \hat{\rho}_4$. So in the roads I_3 and I_4 new shifts $\hat{\xi}_3$ and $\bar{\xi}_4$ are created, where:

$$\begin{cases} \hat{\xi}_3(\rho_3^* - \hat{\rho}_3) = \beta\tilde{\xi}_2(\hat{\rho}_2 - \bar{\rho}_2) = \xi_{3,0}(\rho_3^* - \rho_{3,0}), \\ \bar{\xi}_4(\hat{\rho}_4 - \bar{\rho}_4) = (1-\beta)\tilde{\xi}_2(\hat{\rho}_2 - \bar{\rho}_2) = \frac{1-\beta}{\beta}\xi_{3,0}(\rho_3^* - \rho_{3,0}). \end{cases}$$

Now, if a wave $(\rho_3^*, \bar{\rho}_3)$ with shift $\bar{\xi}_3 = 0$ interacts in I_3 with the wave $(\hat{\rho}_3, \rho_3^*)$ $(\tau(\hat{\rho}_3) < \bar{\rho}_3 < 1)$, then it remains a wave $(\hat{\rho}_3, \bar{\rho}_3)$ with negative speed and with shift $\tilde{\xi}_3$ such that

$$\tilde{\xi}_3(\bar{\rho}_3 - \hat{\rho}_3) = \hat{\xi}_3(\rho_3^* - \hat{\rho}_3) = \xi_{3,0}(\rho_3^* - \rho_{3,0}).$$

If the two waves on road I_4 does non interact, and this happens choosing appropriately the position of waves, then in the road I_4 the L^1–distance is

$$2\frac{1-\beta}{\beta}\left|\xi_{3,0}(\rho_3^* - \rho_{3,0})\right|,$$

and so the lemma is proved. □

Applying repeatedly Lemma 6.3, we produce a counterexample to the Lipschitz continuous dependence by initial data as the next proposition shows.

Proposition 6.1 *Let $C > 0$, let J be a junction and let $(\rho_{1,0}, \ldots, \rho_{4,0})$ be an equilibrium configuration as in Lemma 6.3. There exist two piecewise constant initial data satisfying the equilibrium configuration at J such that their L^1–distance is finite, but the L^1–distance between the corresponding two solutions increases by the multiplication factor C.*

PROOF. Let n be big enough so that

$$\left(1 + 2n\frac{1-\beta}{\beta}\right) > C.$$

We want to define an initial data that provides the desired increase. We choose ρ_3^* and two finite sequences $(\bar{\rho}_2^i)$, $(\bar{\rho}_3^i)$, $i = 1, \ldots, n$, so that, letting $\hat{\rho}_2^i$, $\hat{\rho}_3^i$ be the states determined as in Lemma 6.3, we have:

$$\begin{cases} \rho_3^* \in]\tau(\rho_{3,0}), 1], \\ \bar{\rho}_2^i \in [0, \tau(\hat{\rho}_2^i)[, \quad i = 1, \ldots, n, \\ \bar{\rho}_3^i \in]\tau(\hat{\rho}_3^i), 1], \quad i = 1, \ldots, n. \end{cases}$$

It is easy to check that these sequences can be defined by induction.

The piecewise constant initial data in I_3 is given by

$$\begin{cases} \rho_{3,0}, & \text{if } 0 < x < x^*, \\ \rho_3^*, & \text{if } x^* < x < \hat{x}_1, \\ \tilde{\rho}_3^1, & \text{if } \hat{x}_1 < x < \hat{x}_2, \\ \vdots & \dots \\ \bar{\rho}_3^n, & \text{if } \tilde{x}_n < x, \end{cases}$$

where the values x^*, $\hat{x}_1, \dots, \hat{x}_n$ are to be determined in the sequel. If $\xi_{3,0}$ denotes the shift of the wave $(\rho_{3,0}, \rho_3^*)$ and if no more shifts are present, then the L^1–distance is given by

$$|\xi_{3,0}| \left(\rho_3^* - \rho_{3,0}\right).$$

The initial data on I_2 is

$$\begin{cases} \rho_{2,0}, & \text{if } \tilde{x}_1 < x < 0, \\ \hat{\rho}_2^1, & \text{if } \tilde{x}_2 < x < \tilde{x}_1, \\ \vdots & \dots \\ \hat{\rho}_2^n, & \text{if } x < \tilde{x}_n \\ \vdots & \dots, \end{cases}$$

where $\tilde{x}_1, \dots, \tilde{x}_n$ are to be chosen appropriately.

The speed of the wave $(\rho_{3,0}, \rho_3^*)$ is given by the Rankine–Hugoniot condition

$$\frac{f(\rho_{3,0}) - f(\rho_3^*)}{\rho_{3,0} - \rho_3^*},$$

and consequently the time needed to go to the junction J is

$$\bar{T} = -\frac{\left(\rho_{3,0} - \rho_3^*\right) x^*}{f(\rho_{3,0}) - f(\rho_3^*)}.$$

Clearly we adjust $\bar{T}$, choosing x^*. Applying n times Lemma 6.3 and adjusting the interaction times by choosing appropriately $\bar{x}_i$, $\tilde{x}_i$, $i \in \{1, \dots, n\}$, we deduce that the L^1–distance at $t = T$ is given by

$$\left(1 + 2n\frac{1-\beta}{\beta}\right) |\xi_{3,0}(\rho_3^* - \rho_{3,0})|,$$

which concludes the proof. □

Remark 6.2 The process described in the proof of Proposition 6.1 cannot be infinitely repeated, since the strength of the shifts becomes bigger and bigger as n goes to infinity. Therefore, with this method, it is not possible to prove a blow–up of the L^1–distance in finite time.

In some special cases the Lipschitz continuous dependence holds as we show in the next subsections.

6.1 Network with only one junction and restricted domains.

We consider a road network with only one junction J and with I_1, I_2 incoming roads and I_3, I_4 outgoing roads. We define

$$\mathcal{D} := \left\{\bar{\rho} = (\bar{\rho}_1, \dots, \bar{\rho}_4) \in L^\infty(I_1 \times \dots \times I_4) \cap L^1(I_1 \times \dots \times I_4) : \bar{\rho}_j \in [0, \sigma], j = 3, 4\right\}.$$

The following theorem holds.

Theorem 6.1 *For every $T > 0$, there exists a Lipschitz continuous semigroup $S : [0, T] \times \mathcal{D} \to \mathcal{D}$ so that, for every $\bar{\rho} \in \mathcal{D}$, $\rho(t, x) = S(t, \bar{\rho})(x)$ is an admissible solution such that $\rho(0, x) = \bar{\rho}(x)$.*

Before proving the theorem, we consider the following lemma.

Lemma 6.4 *Let $T > 0$ and let $\rho, \tilde{\rho}$ be two wave front tracking solutions connected by shifts such that $\rho(0, \cdot) \in \mathcal{D}$ and $\tilde{\rho}(0, \cdot) \in \mathcal{D}$. Then, for every $t \in [0, T]$, we have:*

$$\|\rho(t, \cdot) - \tilde{\rho}(t, \cdot)\|_{L^1} = \sum_{\theta \in \Theta(t)} \left|\xi^\theta \Delta \rho^\theta\right| = \|\rho(0, \cdot) - \tilde{\rho}(0, \cdot)\|_{L^1},$$

where $\Theta(t)$ denotes the set of the jumps of $\rho(t, \cdot)$ with shifts.

Proof. We note first that $\mathcal{D}$ is invariant with respect approximate wave front tracking solutions. Since $\rho_j \in [0, \sigma]$ for every $j \in \{3, 4\}$, each wave on I_3 and I_4 has positive speed and so shifts on outgoing roads cannot propagate themselves on other roads. The conclusion easily follows from Lemma 6.2 and Lemma 5.3. □

Proof of Theorem 6.1. For every $T > 0$, by Theorem 5.1, a solution exists for every initial data in $\mathcal{D}$. Fixed $\rho, \tilde{\rho} \in \mathcal{D}$, we denote by $\rho_\nu, \tilde{\rho}_\nu$ two approximate wave front tracking solutions. As in [6, 7], to control the norm $\|\rho_\nu(t, \cdot) - \tilde{\rho}_\nu(t, \cdot)\|_{L^1}$, $t \in [0, T]$, it is enough to control the lengths of the shifts. Therefore, by Lemma 6.4, we obtain

$$\|\rho_\nu(t, \cdot) - \tilde{\rho}_\nu(t, \cdot)\|_{L^1} \le \|\rho_\nu(0, \cdot) - \tilde{\rho}_\nu(0, \cdot)\|_{L^1}$$

for every $t \in [0, T]$. Passing to the limit in the last expression, we obtain the thesis.

6.2 Finite number of big waves and bad data.

Here we want to show a more general result about the Lipschitz continuity with respect to initial data. We prefer to omit the proof of this result, since it can be done with the same techniques as in the last subsection.

Let us consider a road network $(\mathcal{I}, \mathcal{J})$.

Definition 6.1 *Let us fix an approximate wave front tracking solution ρ. For every junction J and for every incoming road I_i, there exists a function $b_\rho(J, i, \cdot)$ defined on $[0, T]$ such that*

$$b_\rho(J, i, t) = \begin{cases} 0, & \text{if } \rho_i(t, b_i-) \in [\sigma, 1], \\ 1, & \text{if } \rho_i(t, b_i-) \in [0, \sigma[. \end{cases}$$

Fixed $T > 0$, we consider a solution ρ defined on $[0, T]$ such that, for every $t \in [0, T]$, $\rho(t, \cdot)$ is a bounded variation function. If ρ_ν is a sequence of approximate wave front tracking solutions (briefly AWFTS), then we say that the sequence ρ_ν has the property (H) if:

1. there exists $M \in \mathbb{N}$ such that the function $b_{\rho_\nu}(J, i, \cdot)$ has at most M discontinuities for every $J \in \mathcal{J}$, for every $i \in \{1, \ldots, N\}$ and for every $\nu \geq 0$;

2. there exists $\delta > 0$ such that
$$|\rho_\nu(t, a_i+) - \sigma| > \delta$$
and
$$|\rho_\nu(t, b_i-) - \sigma| > \delta$$
for every $J \in \mathcal{J}$, for every $i \in \{1, \ldots, N\}$, for every $\nu \geq 0$ and for every $t \in [0, T]$.

The following proposition holds.

Proposition 6.2 *There exists $\eta > 0$ and a Lipschitz continuous semigroup S defined on $[0, T] \times \mathcal{D}_\rho^\eta$, where*

$$\begin{aligned} \mathcal{D}_\rho^\eta := \{\bar\rho \quad : \quad & \exists (\rho_\nu)_{\nu \in \mathbb{N}} \text{ sequence of AWFTS satisfying (H)}, \\ & \rho_\nu(0, \cdot) \to \bar\rho(\cdot) \text{ in } L^1, \text{Tot.Var.}(\rho_\nu(0, \cdot) - \rho(0, \cdot)) < \eta\}. \end{aligned}$$

7 Time Dependent Traffic.

In this section we consider a model of traffic including traffic lights and time dependent traffic. The latter means that the choice of drivers at junctions may depend on the period of the day, for instance during the morning the traffic flows towards some specific parts of the network and during the evening it may flow back. This means that the matrix A may depend on time t.

Consider a single junction J as in Section 3 with two incoming roads I_1, I_2 and two outgoing ones I_3 and I_4. Let $\alpha = \alpha(t)$, $\beta = \beta(t)$ be two piecewise constant functions such that

$$0 < \alpha(t) < 1, \qquad 0 < \beta(t) < 1, \qquad \alpha(t) \neq \beta(t), \tag{7.29}$$

for each $t \geq 0$. Moreover let $\chi_1 = \chi_1(t), \chi_2 = \chi_2(t)$ be piecewise constant maps such that

$$\chi_1(t) + \chi_2(t) = 1, \qquad \chi_i(t) \in \{0, 1\}, \qquad i = 1, 2,$$

for each $t \geq 0$. The two maps represent traffic lights, the value 0 corresponding to red light and the value 1 to green light.

Definition 7.1 *Consider $\rho = \big(\rho_1, ..., \rho_4\big)$ with bounded variation. We say that ρ is a solution at the junction J if it satisfies (i), (iii) of Definition 2.1 and the following property holds:*

(iv) $f(\rho_3(t, a_3+)) = \alpha(t)\chi_1(t)f(\rho_1(t, b_1-)) + \beta(t)\chi_2(t)f(\rho_2(t, b_2-))$ *and* $f(\rho_4(t, a_4+)) = \big(1 - \alpha(t)\big)\chi_1(t)f(\rho_1(t, b_1+)) + \big(1 - \beta(t)\big)\chi_2(t)f(\rho_2(t, b_2+))$ *for each* $t > 0$.

The construction of the solution can be done as in Section 5. However, the total variation of $f(\rho)$ does not depend continuously on the total variation of the maps $\alpha(\cdot)$, $\beta(\cdot)$. Indeed, let us suppose that there are no traffic lights, i.e. $\chi_i \equiv 1$, and let

$$\alpha(t) = \begin{cases} \eta_1 & \text{if } 0 \leq t \leq \bar{t}, \\ \eta_2 & \text{if } \bar{t} \leq t \leq T, \end{cases} \qquad \beta(t) = \begin{cases} \eta_2 & \text{if } 0 \leq t \leq \bar{t}, \\ \eta_1 & \text{if } \bar{t} \leq t \leq T, \end{cases}$$

where $0 < \eta_2 < \eta_1 < \dfrac{1}{2}$ and $0 < \bar{t} < T$. Consider the initial data $(\rho_{1,0},\, \rho_{2,0},\, \rho_{3,0},\, \rho_{4,0})$, where

$$f(\rho_{1,0}) = f(\rho_{4,0}) = f(\sigma), \qquad f(\rho_{2,0}) = f(\rho_{3,0}) = \frac{\eta_1}{1 - \eta_2} f(\sigma),$$

and

$$\sigma < \rho_{2,0} < 1, \qquad 0 < \rho_{3,0} < \sigma.$$

This is an equilibrium configuration and hence the solution of the Riemann Problem for $0 \leq t \leq \bar{t}$. At time $t = \bar{t}$ we have to solve a new Riemann Problem. Let $(\hat{\rho}_1,\, \hat{\rho}_2,\, \hat{\rho}_3,\, \hat{\rho}_4)$ be the new solution. We have:

$$f(\hat{\rho}_2) = f(\hat{\rho}_4) = f(\sigma), \qquad f(\hat{\rho}_1) = f(\hat{\rho}_3) = \frac{\eta_1}{1 - \eta_2} f(\sigma).$$

Now, if $\eta_1 \to \eta_2$, then

$$\text{Tot.Var.}\big(\alpha; [0, T]\big) \longrightarrow 0, \qquad \text{Tot.Var.}\big(\beta; [0, T]\big) \longrightarrow 0,$$

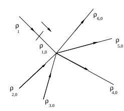

Figure 3: Configuration at J.

but

$$\big(f(\rho_{1,0}),\ f(\rho_{2,0})\big) \longrightarrow \left(f(\sigma),\ \frac{\eta_2}{1-\eta_2} f(\sigma)\right), \qquad \big(f(\hat{\rho}_1),\ f(\hat{\rho}_2)\big) \longrightarrow \left(\frac{\eta_2}{1-\eta_2} f(\sigma),\ f(\sigma)\right),$$

hence $\mathrm{Tot.Var.}\big(f(\rho);[0,T]\big)$ is bounded away from zero.

*

A Appendix: Total Variation of the Fluxes.

In this section we show an example in which the total variation of the flux increases due to interactions of waves with junctions.

Consider a single junction with three incoming roads and three outgoing ones, the matrix

$$A \doteq \begin{pmatrix} 1/2 & 1/2 & 1/3 \\ 1/3 & 1/2 & 1/2 \\ 1/6 & 0 & 1/6 \end{pmatrix} \quad \text{condition (C)??} \tag{A.30}$$

and the constants $\rho_1,\ \rho_{1,0}, ...,\ \rho_{6,0} \in [0,1]$ such that

$$\rho_{1,0} = \rho_{3,0} = \rho_{4,0} = \rho_{5,0} = \sigma, \qquad \sigma < \rho_{2,0} < 1,$$

$$0 < \rho_{6,0},\ \rho_1 < \sigma, \qquad f(\rho_{2,0}) = \frac{1}{3},\ \ f(\rho_{6,0}) = \frac{1}{3}.$$

Assume that $f(\sigma) = 1$, then $(\rho_{1,0}, ...,\ \rho_{6,0})$ is an equilibrium configuration and ρ given by

$$\rho_1(0,x) = \begin{cases} \rho_{1,0} & \text{if } x_1 \le x \le b_1, \\ \rho_1 & \text{if } x < x_1, \end{cases} \qquad \rho_i(0,\cdot) \equiv \rho_{i,0}, \quad i = 2, ..., 6,$$

is a solution. Moreover, the plane

$$\frac{1}{6}\gamma_1 + \frac{1}{6}\gamma_3 = 1$$

does not intersect the cube $[0,1]^3$ and the point $\big(f(\rho_{1,0}), ..., f(\rho_{6,0})\big)$ is on the intersection of the planes

$$\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{3}\gamma_3 = 1, \qquad \frac{1}{3}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\gamma_3 = 1,$$

that is the line described by the map

$$\gamma_1 \mapsto \left(\gamma_1, 2 - \frac{5}{3}\gamma_1, \gamma_1\right). \tag{A.31}$$

At some time, say $\bar{t}$, the wave $(\rho_1, \rho_{1,0})$ interacts with the junction. Let $(\hat{\rho}_1, ..., \hat{\rho}_6)$ be the solution of the Riemann Problem at the junction for the data $(\rho_1,\ \rho_{2,0}, ...,\ \rho_{6,0})$. Since the map E increases on the line described by (A.31), the point $\big(f(\hat{\rho}_1), ..., f(\hat{\rho}_6)\big)$ is on the curve (A.31) and

$$\begin{aligned} f(\hat{\rho}_1) &= f(\hat{\rho}_3) = f(\rho_1), & f(\hat{\rho}_2) &= 2 - \tfrac{5}{3} f(\rho_1), \\ f(\hat{\rho}_4) &= f(\hat{\rho}_5) = f(\sigma), & f(\hat{\rho}_6) &= \tfrac{1}{3} f(\rho_1). \end{aligned}$$

We get

$$\text{Tot.Var.}\big(f(\rho(\bar{t}-, \cdot))\big) = 1 - f(\rho_1),$$

while

$$\text{Tot.Var.}\big(f(\rho(\bar{t}+, \cdot))\big) = 4\big(1 - f(\rho_1)\big) > \text{Tot.Var.}\big(f(\rho(\bar{t}-, \cdot))\big).$$

B Appendix: Total Variation of the Densities.

Consider a junction J with two incoming roads and two outgoing ones that we parameterize with the intervals $]-\infty,\ b_1]$, $]-\infty,\ b_2]$, $[a_3,\ +\infty[$, $[a_4,\ +\infty[$ respectively. We suppose that $0 < \beta < \alpha < 1/2$, where α and β are the entries of the matrix A as in (5.16).
Define a solution ρ by

$$\rho_1(0,x) = \begin{cases} \rho_{1,0} & \text{if } x_1 \le x \le b_1, \\ \rho_1 & \text{if } x < x_1, \end{cases} \qquad \rho_2(0,x) = \rho_{2,0}, \quad \rho_3(0,x) = \rho_{3,0}, \quad \rho_4(0,x) = \rho_{4,0}, \tag{B.32}$$

where $\rho_1,\ \rho_{1,0},\ \rho_{2,0},\ \rho_{3,0},\ \rho_{4,0}$ are constants such that

$$\sigma < \rho_{2,0} < 1, \quad \sigma < \rho_{3,0} < 1, \quad 0 \le \rho_1 < \sigma, \quad \rho_{1,0} = \rho_{4,0} = \sigma, \tag{B.33}$$

$$f(\rho_{1,0}) = f(\rho_{4,0}) = f(\sigma), \quad f(\rho_{2,0}) = f(\rho_{3,0}) = \frac{\alpha}{1-\beta} f(\sigma),$$

so $(\rho_{1,0},\ \rho_{2,0},\ \rho_{3,0},\ \rho_{4,0})$ is an equilibrium configuration.

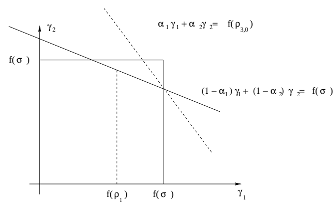

Figure 4: Solution to the Riemann problem at J.

After some time the wave $(\rho_1, \rho_{1,0})$ interacts with the junction. Let $(\hat{\rho}_1,\ \hat{\rho}_2,\ \hat{\rho}_3,\ \hat{\rho}_4)$ be the solution of the Riemann Problem in the junction for the data $(\rho_1,\ \rho_{2,0},\ \rho_{3,0},\ \rho_{4,0})$, see Figure 4. By (B.32) and (B.33),

$$f(\hat{\rho}_1) = f(\rho_1), \quad f(\hat{\rho}_2) = \frac{f(\sigma) - (1-\alpha) f(\rho_1)}{1-\beta},$$

$$f(\hat{\rho}_3) = \frac{\alpha - \beta}{1-\beta} f(\rho_1) + \frac{\beta}{1-\beta} f(\sigma), \quad f(\hat{\rho}_4) = f(\sigma),$$

and

$$0 < \hat{\rho}_3 < \sigma \leq \hat{\rho}_2 < 1. \tag{B.34}$$

Therefore, if $\rho_1 \to \rho_{1,0} = \sigma$, then

$$f(\hat{\rho}_3) \longrightarrow \frac{\alpha}{1-\beta} f(\sigma) = f(\rho_{3,0}),$$

and, by (B.34), (B.33), we have $\hat{\rho}_3 \to \tau(\rho_{3,0})$. Then, we are able to create on the third road a wave with strength bounded away from zero using an arbitrarily small wave on the first one.

Acknowledgments

The authors would like to thank Prof. Rinaldo M. Colombo for useful discussions.

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